Conic sections

In this chapter we examine the Cartesian graph of any equation:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

In which A,B, and C are not all zero, and show that it is nearly always a conic section. Also, we will give geometric definitions of a circle, parabola, ellipse, and hyperbola and derive their standard equations.

1)The circle: the set of points in a plane whose distance from some fixed center point is a constant radius value. If the center (h,k) and the radius is r, the standard equation for the circle is $(x-h)^2 + (y-k)^2 = r^2$.

2) A parabola is the set of points in a plane that are equidistant from given fixed point (focus) and fixed line (directrix) in the plane.

Table of standard-form

	Equation	Focus	Directrix	Vertex	Opens
1	$y^2 = 4px$	(p,0)	X=-p	(0,0)	To the right
	$(y-k)^2 = 4p(x-h)$	(h+p,k)	X=h-p	(h,k)	To the right
2	$y^2 = -4px$	(-p,0)	X=p	(0,0)	To the left
	$(y-k)^2 = -4p(x-h)$	(h-p,k)	X=h+p	(h,k)	To the left
3	$x^2 = 4py$	(0,p)	y=-p	(0,0)	Up
	$(x-h)^2 = 4p(y-k)$	(h,k+p)	y=k-p	(h,k)	Up
4	$x^2 = -4py$	(0,-p)	y=p	(0,0)	Down
	$(x-h)^2 = -4p(y-k)$	(h,k-p)	y=k+p	(h,k)	Down

Examples: Find the focus, vertex, and directrix of the parabolas and sketch the parabola:

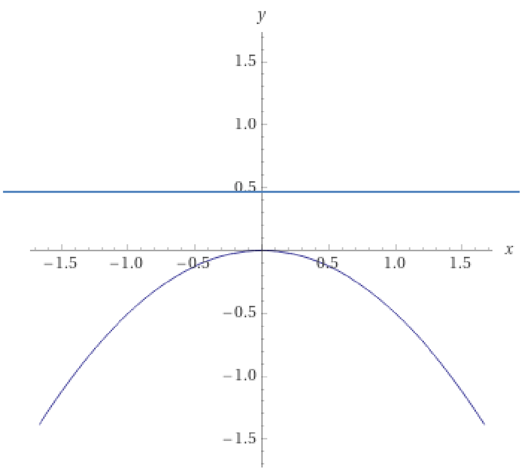
1)$x^2 + 2y = 0$
2)$y^2 - 2y - 12x - 23 = 0$
3) $x^2 + 8y - 4 = 0$

Solution:

1) We find the value of p in the standard equation: $x^2 = -4py$

$$x^2 = -4py \Rightarrow x^2 = -2y \text{ so } 4p = 2 \Rightarrow p = \frac{1}{2}$$

Then, the focus $\left(0, -\frac{1}{2}\right)$, vertex $(0,0)$, and directrix $y = \frac{1}{2}$



2) H.W 3) H.W

3) Ellipse is the set of points in a plane whose distance from two fixed points (foci) in the plane have a constant sum (2a).

Table of standard-form

	Equation	Foci	Vertices	Minor axis	center
1	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$F_{1,2}(\mp c, 0)$ $c^2 = a^2 - b^2$	$A_{1,2}(\mp a, 0)$	$B_{1,2}(0, \mp b)$	(0,0)
	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	$F_{1,2}(h \mp c, k)$	$A_{1,2}(h \mp a, k)$	$B_{1,2}(h, k \mp b)$	(h,k)
2	$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$	$F_{1,2}(0, \mp c)$ $c^2 = a^2 - b^2$	$A_{1,2}(0, \mp a)$	$B_{1,2}(\mp b, 0)$	(0,0)
	$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$	$F_{1,2}(h, k \mp c)$	$A_{1,2}(h, k \mp a)$	$B_{1,2}(h \mp b, k)$	(h,k)

Examples: Find the center, vertices, and foci of the ellipse and sketch the ellipse:

1) $\frac{x^2}{16} + \frac{y^2}{9} = 1$

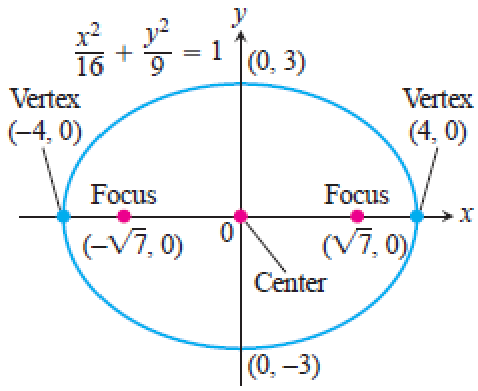
2) $9x^2 + 4y^2 + 36x - 8y + 4 = 0$

3) $x^2 + 10x + 25y^2 = 0$

Solution:

1) Center : (0,0) vertices: $A_{1,2}(\mp a, 0) = A_{1,2}(\mp 4,0)$

Foci: $c^2 = a^2 - b^2 \Rightarrow c = \mp\sqrt{16-9} \Rightarrow c = \mp\sqrt{7} \Rightarrow F_{1,2}(\mp c, 0) = F_{1,2}(\mp\sqrt{7}, 0)$



2)H.W 3)H.W

3) Hyperbola is the set of points in a plane whose distance from two fixed points (foci) in the plane have a constant difference.

Table of standard-form

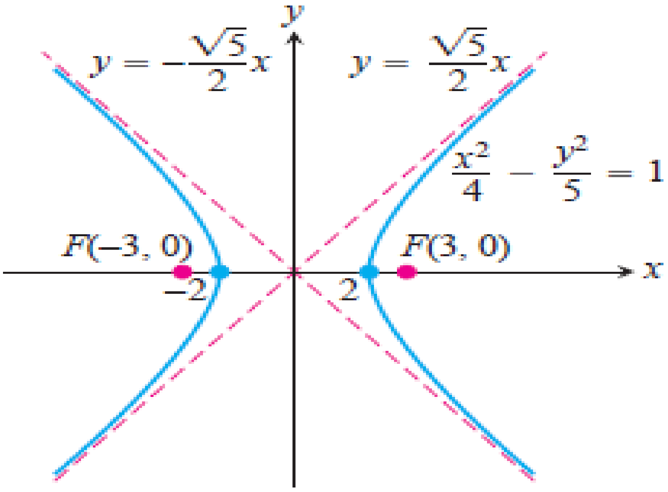
	Equation	Foci	Vertices	Minor axis	center
1	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$F_{1,2}(\mp c, 0)$ $c^2 = a^2 + b^2$	$A_{1,2}(\mp a, 0)$	$B_{1,2}(0, \mp b)$	(0,0)
	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$F_{1,2}(h \mp c, k)$	$A_{1,2}(h \mp a, k)$	$B_{1,2}(h, k \mp b)$	(h,k)
2	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$	$F_{1,2}(0, \mp c)$ $c^2 = a^2 + b^2$	$A_{1,2}(0, \mp a)$	$B_{1,2}(\mp b, 0)$	(0,0)
	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$	$F_{1,2}(h, k \mp c)$	$A_{1,2}(h, k \mp a)$	$B_{1,2}(h \mp b, k)$	(h,k)

4) Examples: Find the center, vertices, and foci of the hyperbola and sketch the hyperbola: (i) $\frac{x^2}{4} - \frac{y^2}{5} = 1$ (ii) $4x^2 - y^2 + 8x + 2y - 1 = 0$.

Solution:

1) Center : $(0,0)$ vertices: $A_{1,2}(\mp a, 0) = A_{1,2}(\mp 2,0)$

Foci: $c^2 = a^2 + b^2 \Rightarrow c = \mp\sqrt{4+5} \Rightarrow c = \mp\sqrt{9} \Rightarrow F_{1,2}(\mp c, 0) = F_{1,2}(\mp 3,0)$

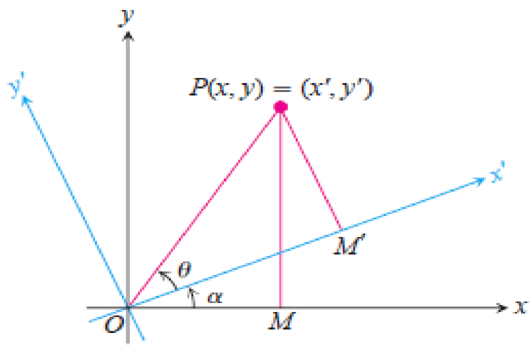


2)H.W

Note that eccentricity is $\frac{c}{a}$ $(e = \frac{c}{a})$ where $0 < e < 1$.

Equations for Rotating coordinate Axes

The equations for the rotations we use are derived in the following way. In the notation of Figure (1) which shows in anticlockwise rotation about the origin through an angle ($\propto$),



$\cos(\propto +\theta) = \frac{x}{\overrightarrow{op}}$, $\sin(\propto +\theta) = \frac{y}{\overrightarrow{op}}$

Figure (1)

$x = \overrightarrow{op}\cos(\propto)\cos(\theta) - \overrightarrow{op}\sin(\propto)\sin(\theta)$.

$y = \overrightarrow{op}\sin(\propto)\cos(\theta) + \overrightarrow{op}\cos(\propto)\sin(\theta)$.

Note that $x' = \overrightarrow{op}\cos(\theta)$ *and* $y' = \overrightarrow{op}\sin(\theta)$

Thus,

$x = x'\cos(\propto) - y'\sin(\propto)$.

$y = x'\sin(\propto) + y'\cos(\propto)$.

Now, if we apply equations above to the equation $\boldsymbol{Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0}$,

We obtain a equation $A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = \mathbf{0}$, where:

$A' = A\cos^2\alpha + B\cos\alpha\sin\alpha + C\sin^2\alpha$

$B' = B\cos 2\alpha + (C - A)\sin 2\alpha$

$C' = A\sin^2\alpha - B\cos\alpha\sin\alpha + C\cos^2\alpha$

$D' = D\cos\alpha + E\sin\alpha$

$E' = E\cos\alpha - D\sin\alpha$

$F' = F$.

To find ($\propto$), we put $B' = 0$ and solve the resulting equation,

$Bcos2\alpha + (C - A)sin2\alpha = 0$, so $\cot 2\alpha = \frac{A-C}{B}$ or $\tan 2\alpha = \frac{B}{A-C}$

EXAMPLE: Decided whether the conic section with following equations represents a Parabola, an Ellipse, or Hyperbola.

i)$xy = 2$ (ii)$2x^2 + \sqrt{3}xy + y^2 = 10$

(iii) $3x^2 + 2\sqrt{3}xy + y^2 - 8x + 8\sqrt{3}y = 0$

Solution:

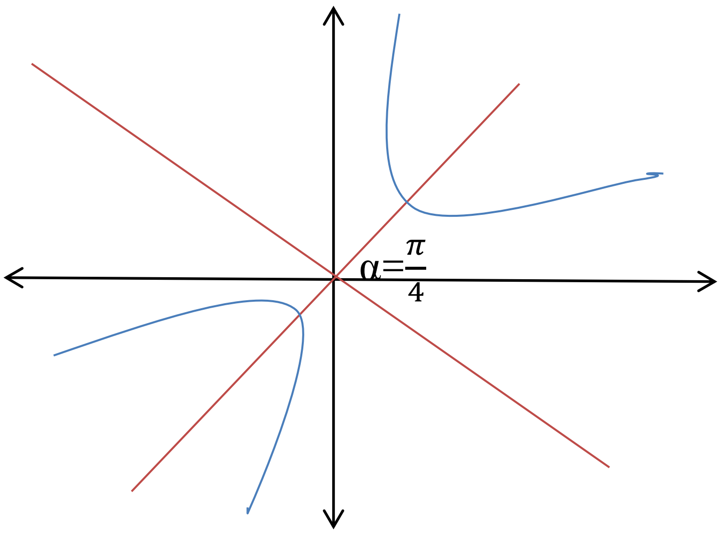
i) The equation $xy = 2$ has $A = 0, B = 1, and\ C = 0$. We substitute these values into $tan\ 2\alpha = \frac{B}{A-C} \rightarrow\ tan\ 2\alpha = \frac{1}{0-0} \rightarrow 2\alpha = tan^{-1}\frac{1}{0} \rightarrow 2\alpha = \frac{\pi}{2} \rightarrow \alpha = \frac{\pi}{4}$

Thus,

$$x = x'\cos(\propto) - y'\sin(\propto) \rightarrow x = \frac{x' - y'}{\sqrt{2}}.$$

$$y = x'\sin(\propto) + y'\cos(\propto) \rightarrow y = \frac{x' + y'}{\sqrt{2}}.$$

$$\boldsymbol{xy = 2 \rightarrow \left(\frac{x'-y'}{\sqrt{2}}\right)\left(\frac{x'+y'}{\sqrt{2}}\right) = 2 \rightarrow x'^2 - y'^2 = 4 \rightarrow \frac{x'^2}{4} - \frac{y'^2}{4} = 1}$$



ii)H.W iii) H.W

10.5 Polar Coordinates

In this section, we study polar coordinates and their relation to Cartesian coordinates. While a point in the plane has just one pair of Cartesian coordinates, it has infinitely many pairs of polar coordinates. This has interesting consequences for graphing, as we will see in the next section.

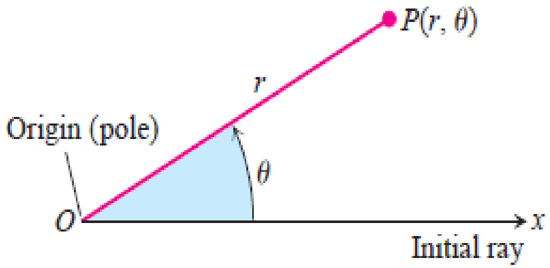


FIGURE 10.35 To define polar coordinates for the plane, we start with an origin, called the pole, and an initial ray.

Definition of Polar Coordinates

To define polar coordinates, we first fix an **origin** O (called the **pole**) and an **initial ray** from O (Figure 10.35). Then each point P can be located by assigning to it a **polar coordinate pair** (r, θ) in which r gives the directed distance from O to P and θ gives the directed angle from the initial ray to ray OP.

Polar Coordinates

$$P(r, \theta)$$

Directed distance from O to P

Directed angle from initial ray to OP

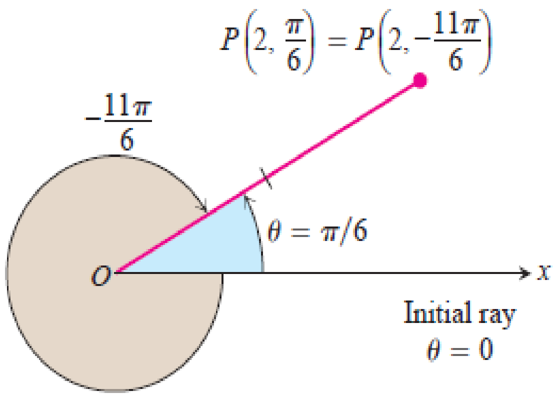


FIGURE 10.36 Polar coordinates are not unique.

As in trigonometry, θ is positive when measured counterclockwise and negative when measured clockwise. The angle associated with a given point is not unique. For instance, the point 2 units from the origin along the ray $\theta = \pi/6$ has polar coordinates $r = 2$, $\theta = \pi/6$. It also has coordinates $r = 2$, $\theta = -11\pi/6$ (Figure 10.36). There are occasions when we wish to allow r to be negative. That is why we use directed distance in defining $P(r, \theta)$. The point $P(2, 7\pi/6)$ can be reached by turning $7\pi/6$ radians counterclockwise from the initial ray and going forward 2 units (Figure 10.37). It can also be reached by turning $\pi/6$ radians counterclockwise from the initial ray and going *backward* 2 units. So the point also has polar coordinates $r = -2$, $\theta = \pi/6$.

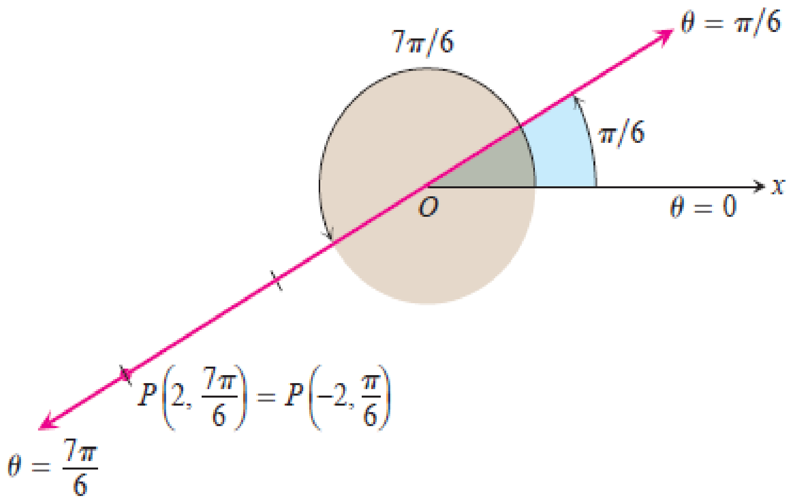


FIGURE 10.37 Polar coordinates can have negative r-values.

EXAMPLE 1 Finding Polar Coordinates

Find all the polar coordinates of the point $P(2, \pi/6)$.

Solution We sketch the initial ray of the coordinate system, draw the ray from the origin that makes an angle of $\pi/6$ radians with the initial ray, and mark the point $(2, \pi/6)$ (Figure 10.38). We then find the angles for the other coordinate pairs of P in which $r = 2$ and $r = -2$.

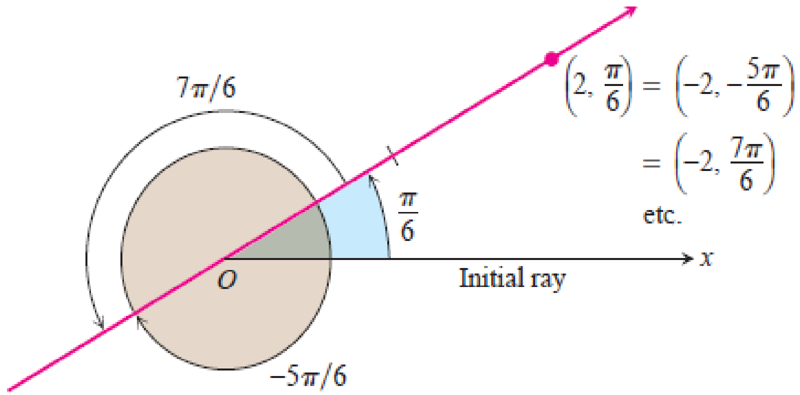


FIGURE 10.38 The point $P(2, \pi/6)$ has infinitely many polar coordinate pairs (Example 1).

For $r = 2$, the complete list of angles is

$$\frac{\pi}{6}, \quad \frac{\pi}{6} \pm 2\pi, \quad \frac{\pi}{6} \pm 4\pi, \quad \frac{\pi}{6} \pm 6\pi, \quad \ldots .$$

For $r = -2$, the angles are

$$-\frac{5\pi}{6}, \quad -\frac{5\pi}{6} \pm 2\pi, \quad -\frac{5\pi}{6} \pm 4\pi, \quad -\frac{5\pi}{6} \pm 6\pi, \quad \ldots .$$

The corresponding coordinate pairs of P are

$$\left(2, \frac{\pi}{6} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \ldots$$

and

$$\left(-2, -\frac{5\pi}{6} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \ldots .$$

When $n = 0$, the formulas give $(2, \pi/6)$ and $(-2, -5\pi/6)$. When $n = 1$, they give $(2, 13\pi/6)$ and $(-2, 7\pi/6)$, and so on. ■

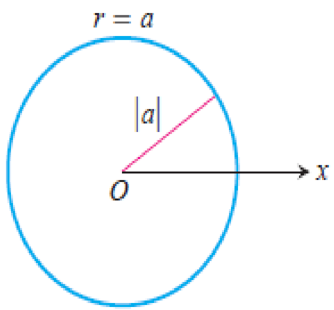


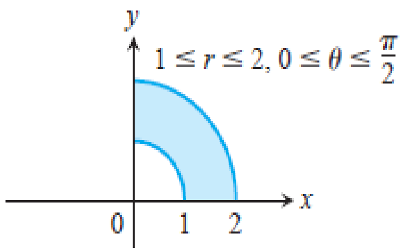
FIGURE 10.39 The polar equation for a circle is $r = a$.

Polar Equations and Graphs

If we hold r fixed at a constant value $r = a \neq 0$, the point $P(r, \theta)$ will lie $|a|$ units from the origin O. As θ varies over any interval of length 2π, P then traces a circle of radius $|a|$ centered at O (Figure 10.39).

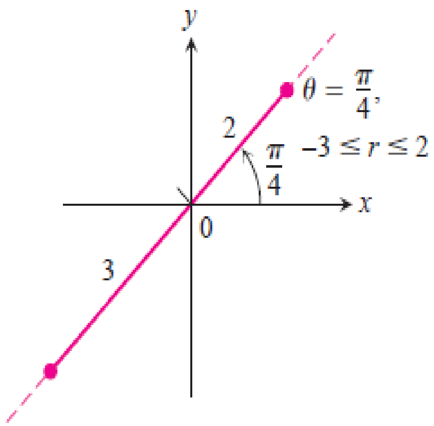
If we hold θ fixed at a constant value $\theta = \theta_0$ and let r vary between $-\infty$ and ∞, the point $P(r, \theta)$ traces the line through O that makes an angle of measure θ_0 with the initial ray.

(a)

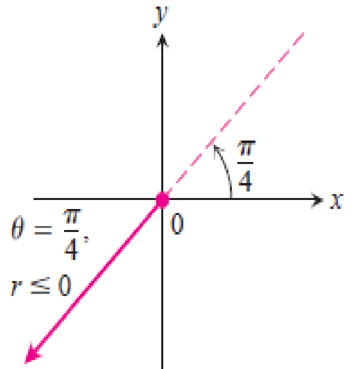


Equation	Graph
$r = a$	Circle radius $|a|$ centered at O
$\theta = \theta_0$	Line through O making an angle θ_0 with the initial ray

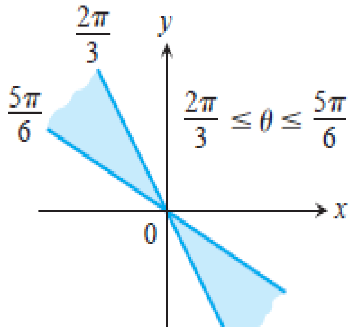
(b)



(c)



(d)



EXAMPLE 2 Finding Polar Equations for Graphs

(a) $r = 1$ and $r = -1$ are equations for the circle of radius 1 centered at O.

(b) $\theta = \pi/6$, $\theta = 7\pi/6$, and $\theta = -5\pi/6$ are equations for the line in Figure 10.38. ■

Equations of the form $r = a$ and $\theta = \theta_0$ can be combined to define regions, segments, and rays.

EXAMPLE 3 Identifying Graphs

Graph the sets of points whose polar coordinates satisfy the following conditions.

(a) $1 \le r \le 2$ and $0 \le \theta \le \dfrac{\pi}{2}$

(b) $-3 \le r \le 2$ and $\theta = \dfrac{\pi}{4}$

(c) $r \le 0$ and $\theta = \dfrac{\pi}{4}$

(d) $\dfrac{2\pi}{3} \le \theta \le \dfrac{5\pi}{6}$ (no restriction on r)

Solution The graphs are shown in Figure 10.40. ■

Relating Polar and Cartesian Coordinates

When we use both polar and Cartesian coordinates in a plane, we place the two origins together and take the initial polar ray as the positive x-axis. The ray $\theta = \pi/2, r > 0$, becomes the positive y-axis (Figure 10.41). The two coordinate systems are then related by the following equations.

Equations Relating Polar and Cartesian Coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad x^2 + y^2 = r^2$$

The first two of these equations uniquely determine the Cartesian coordinates x and y given the polar coordinates r and θ. On the other hand, if x and y are given, the third equation gives two possible choices for r (a positive and a negative value). For each selection, there is a unique $\theta \in [0, 2\pi)$ satisfying the first two equations, each then giving a polar coordinate representation of the Cartesian point (x, y). The other polar coordinate representations for the point can be determined from these two, as in Example 1.

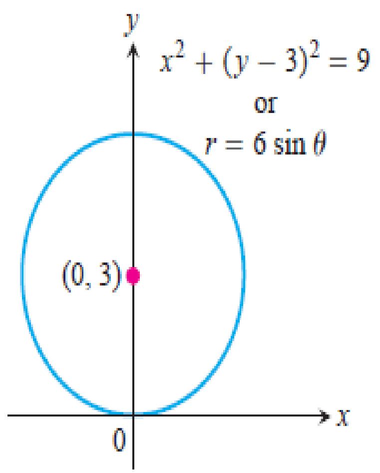
EXAMPLE 4 Equivalent Equations

Polar equation	Cartesian equivalent
$r\cos\theta = 2$	$x = 2$
$r^2\cos\theta\sin\theta = 4$	$xy = 4$
$r^2\cos^2\theta - r^2\sin^2\theta = 1$	$x^2 - y^2 = 1$
$r = 1 + 2r\cos\theta$	$y^2 - 3x^2 - 4x - 1 = 0$
$r = 1 - \cos\theta$	$x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$

With some curves, we are better off with polar coordinates; with others, we aren't. ■

EXAMPLE 5 Converting Cartesian to Polar

Find a polar equation for the circle $x^2 + (y-3)^2 = 9$ (Figure 10.42).



Solution

$$
\begin{aligned}
x^2 + y^2 - 6y + 9 &= 9 && \text{Expand } (y-3)^2. \\
x^2 + y^2 - 6y &= 0 && \text{The 9's cancel.} \\
r^2 - 6r\sin\theta &= 0 && x^2 + y^2 = r^2 \\
r = 0 \quad \text{or} \quad r - 6\sin\theta &= 0 \\
r &= 6\sin\theta && \text{Includes both possibilities}
\end{aligned}
$$

EXAMPLE 6 Converting Polar to Cartesian

Replace the following polar equations by equivalent Cartesian equations, and identify their graphs.

(a) $r \cos \theta = -4$

(b) $r^2 = 4r \cos \theta$

(c) $r = \dfrac{4}{2 \cos \theta - \sin \theta}$

Solution We use the substitutions $r \cos \theta = x$, $r \sin \theta = y$, $r^2 = x^2 + y^2$.

(a) $r \cos \theta = -4$

The Cartesian equation:

$$r \cos \theta = -4$$
$$x = -4$$

The graph: Vertical line through $x = -4$ on the x-axis

(b) $r^2 = 4r \cos \theta$

The Cartesian equation:

$$r^2 = 4r \cos \theta$$
$$x^2 + y^2 = 4x$$
$$x^2 - 4x + y^2 = 0$$
$$x^2 - 4x + 4 + y^2 = 4 \qquad \text{Completing the square}$$
$$(x - 2)^2 + y^2 = 4$$

The graph: Circle, radius 2, center $(h, k) = (2, 0)$

Graphing in Polar Coordinates

This section describes techniques for graphing equations in polar coordinates.

Symmetry

Figure 10.43 illustrates the standard polar coordinate tests for symmetry.

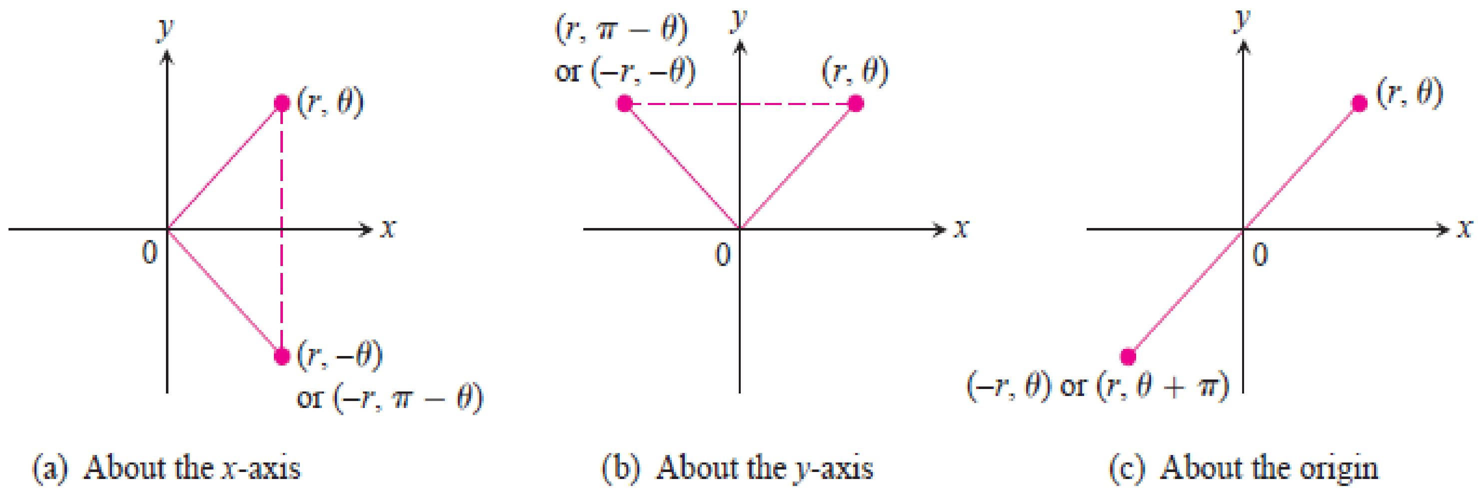


FIGURE 10.43 Three tests for symmetry in polar coordinates.

Symmetry Tests for Polar Graphs

1. *Symmetry about the x-axis:* If the point (r, θ) lies on the graph, the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph (Figure 10.43a).
2. *Symmetry about the y-axis:* If the point (r, θ) lies on the graph, the point $(r, \pi - \theta)$ or $(-r, -\theta)$ lies on the graph (Figure 10.43b).
3. *Symmetry about the origin:* If the point (r, θ) lies on the graph, the point $(-r, \theta)$ or $(r, \theta + \pi)$ lies on the graph (Figure 10.43c).

Graphs of Polar Equations

1)The polar equation is a circle equation if

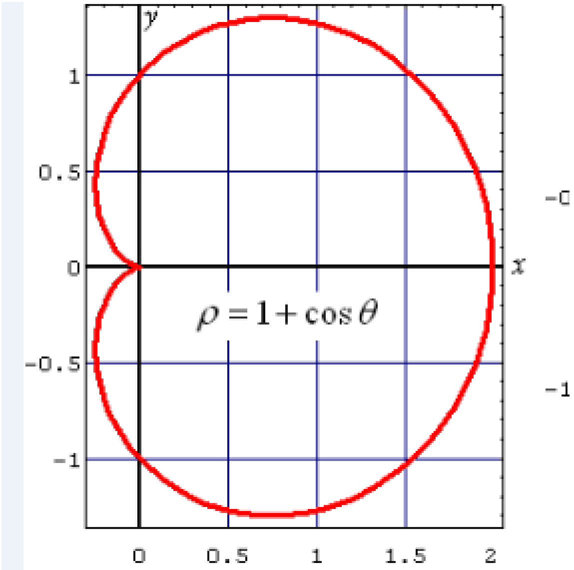
- $r = \mp a$, where $a \neq 0$
- $r = \mp a \sin\theta$, where $a, b \in R - \{0\}$
- $r = \mp a \cos\theta$, where $a, b \in R - \{0\}$
- $r^2 = \mp a \cos\theta$ or • $r^2 = \mp a \sin\theta$, where $a, b \in R - \{0\}$(a semi-circle)

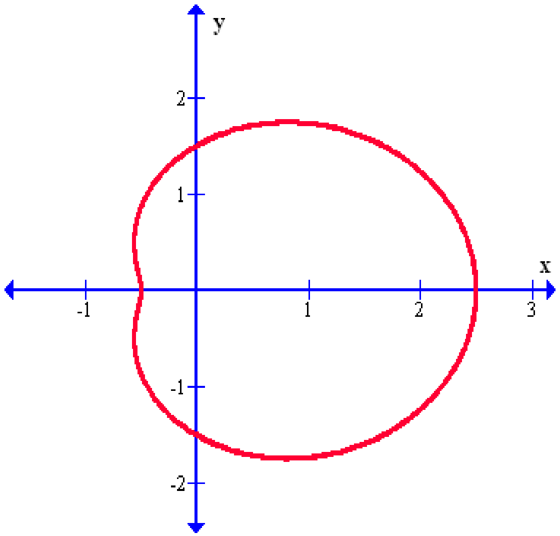
2) **Limaçons has f**ormed as $r = a \pm b\cos\theta$ or $r = a \pm b\sin\theta \quad where\ a, b \in R - \{0\}$**:**

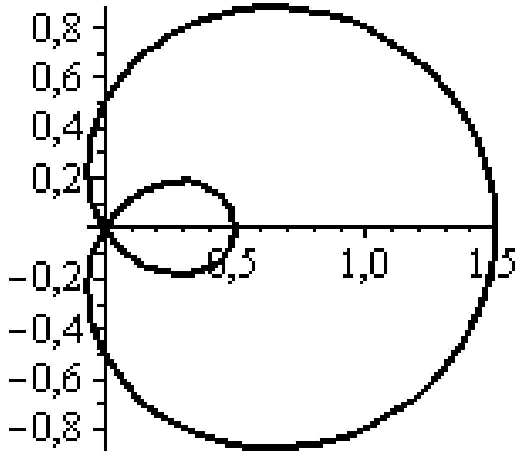
●The polar equation is a ***Cardioid curve*** if $a = b$

●The polar equation is a ***dimpled curve*** if $a > b$

●The polar equation is a ***inner*** loop ***curve*** if $a < b$





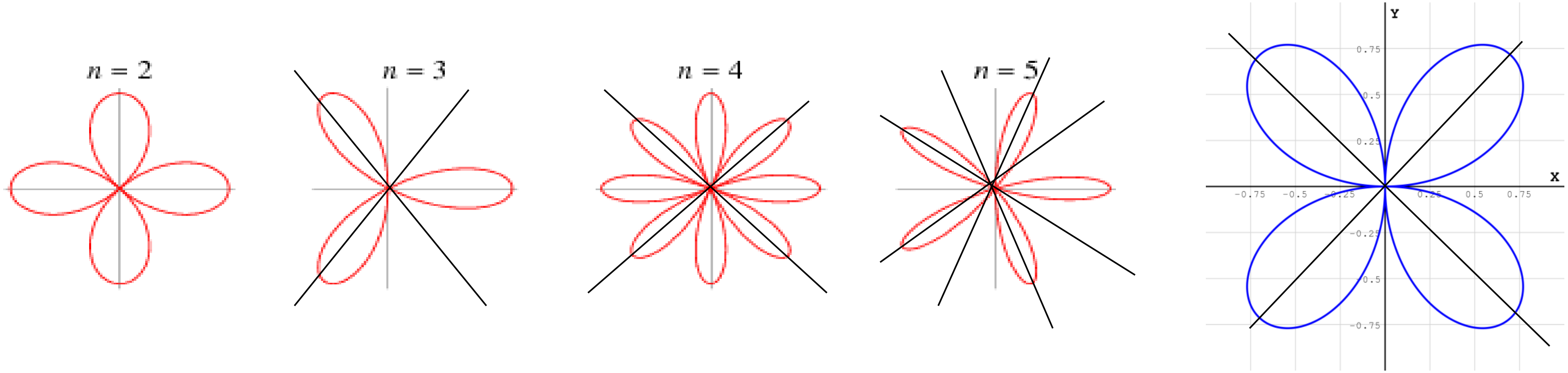


Cardioid curve ***dimpled curve*** ***inner*** loop ***curve***

3) *Rose curve* if the polar equation has form as

- $r = a\cos n\theta$ or $r = a\sin n\theta$ $where\ a \in R - \{0\}\ and\ n \neq 1\ and\ n \in N$

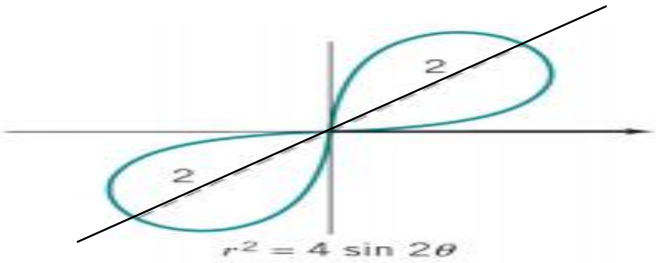
Note that: if n is an odd number then the number of leaves equal n.
If n is an even number then the number of leaves equal $2n$.

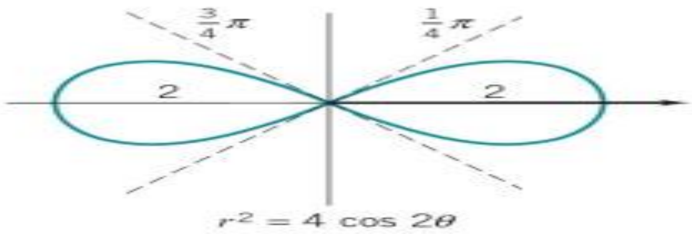


$r = \cos 2\theta$ $r = \cos 3\theta$ $r = \cos 4\theta$ $r = \cos 5\theta$ $r = \sin 2\theta$

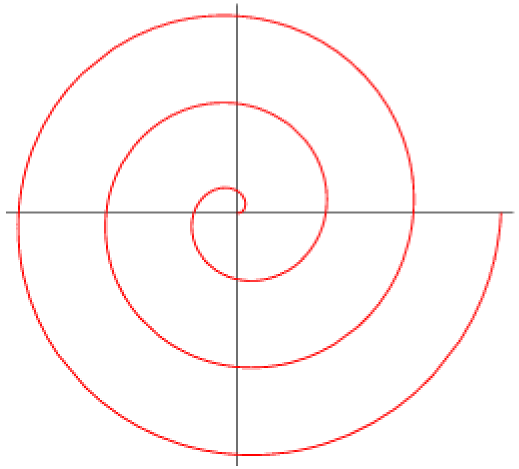
4) ***Lemniscate Curve*** if the polar equation has form as

$r^2 = a\cos n\theta$ or $r^2 = a\sin n\theta$ $where\ a \in R - \{0\}\ and\ n \neq 1\ and\ n \in N$



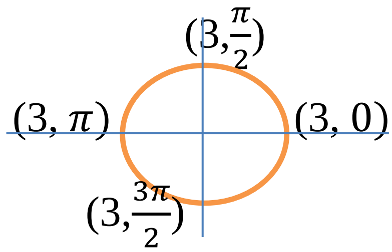


5) The polar equation has ***Spiral curve*** form if $r = a\theta$, $where\ a \in R - \{0\}$



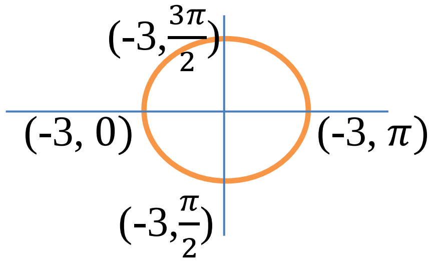
$$r = \theta$$

1) Graph the Curve $r = 3$



Solution: Circle radius 3 centered at $(0, \theta)$

2) Graph the Curve $r = -3$



Solution: Circle radius 3 centered at $(0, \theta)$

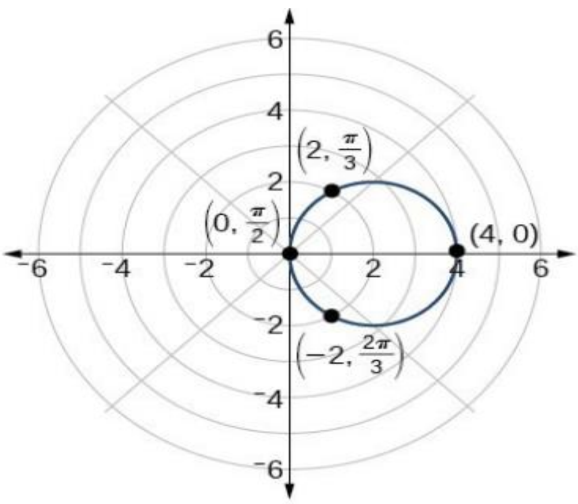
3) Graph the Curve $r = 4\cos\theta$

Solution: The curve is symmetric about the x-axis because (r, θ) on the graph then $r = 4\cos(-\theta) \rightarrow r = 4\cos\theta \rightarrow (r, -\theta)$ on the graph

There is not symmetric about the y-axis and the origin point

θ	r	(r, θ)
0	4	$(4,0)$
$\frac{\pi}{6}$	$2\sqrt{3}$	$\left(3.4, \frac{\pi}{6}\right)$
$\frac{\pi}{4}$	$2\sqrt{2}$	$\left(2.8, \frac{\pi}{4}\right)$
$\frac{\pi}{3}$	2	$\left(2, \frac{\pi}{3}\right)$
$\frac{\pi}{2}$	0	$\left(0, \frac{\pi}{2}\right)$
π	-4	$(-4, \pi)$

$r = 4\cos\theta$



4) Graph the Curve $r = 4\sin\theta$

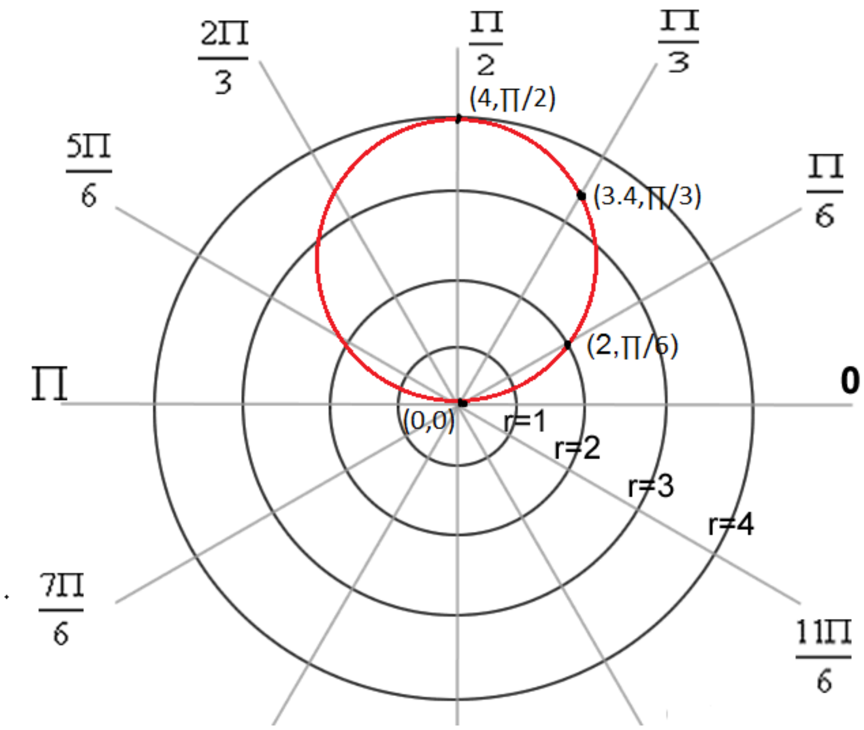
Solution: The curve is symmetric about the *y*-axis because (r,θ) on the graph then

$-r = 4\sin(-\theta) \rightarrow r = 4\sin\theta \rightarrow (-r,-\theta)$ on the graph

There is not symmetric about the *x*-axis and the origin point

θ	r	(r,θ)
0	0	$(0,0)$
$\frac{\pi}{6}$	2	$\left(2,\frac{\pi}{6}\right)$
$\frac{\pi}{4}$	$2\sqrt{2}$	$\left(2.8,\frac{\pi}{4}\right)$
$\frac{\pi}{3}$	$2\sqrt{3}$	$\left(3.4,\frac{\pi}{3}\right)$
$\frac{\pi}{2}$	4	$\left(4,\frac{\pi}{2}\right)$

$r = 4\sin\theta$



5) Graph the Curve $r^2 = 4\cos\theta$

Solution:

●The curve is symmetric about the x-axis because (r,θ) on the graph then

$$r^2 = 4\cos(-\theta) \rightarrow r^2 = 4\cos\theta \rightarrow (r,-\theta) \text{ on the graph}$$

●The curve is symmetric about the y-axis because (r,θ) on the graph then

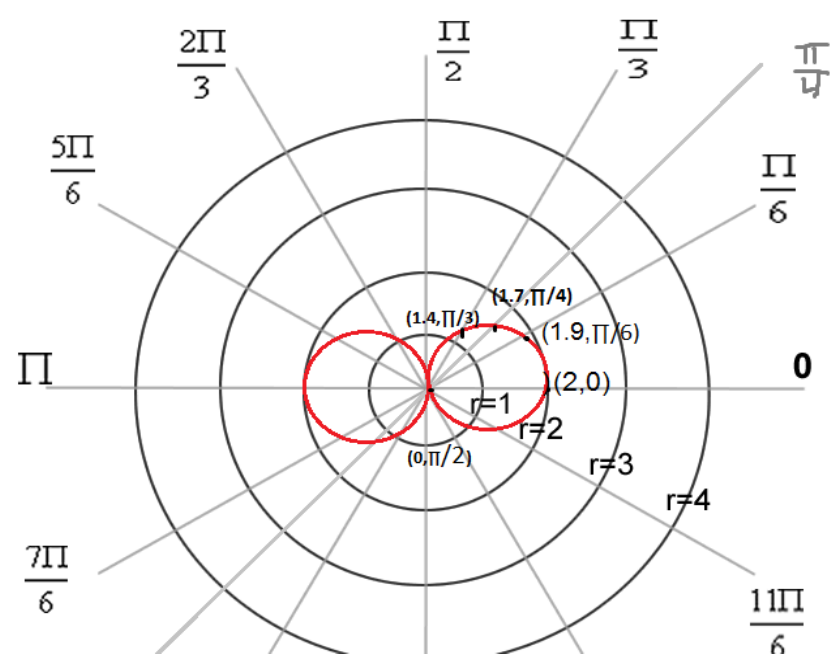
$(-r)^2 = 4\cos(-\theta) \rightarrow r^2 = 4\cos\theta \rightarrow (-r,-\theta)$ on the graph

●The curve is symmetric about the origin point because (r,θ) on the graph then

$(-r)^2 = 4\cos(\theta) \rightarrow r^2 = 4\cos\theta \rightarrow (-r,\theta)$ on the graph

θ	r^2	r	(r,θ)
0	4	± 2	$(\pm 2,0)$
$\frac{\pi}{6}$	$\frac{4\sqrt{3}}{2}$	± 1.9	$\left(\pm 1.9,\frac{\pi}{6}\right)$
$\frac{\pi}{4}$	$\frac{4}{\sqrt{2}}$	± 1.7	$\left(\pm 1.7,\frac{\pi}{4}\right)$
$\frac{\pi}{3}$	$\frac{4}{2}$	± 1.4	$\left(\pm 1.4,\frac{\pi}{3}\right)$
$\frac{\pi}{2}$	0	0	$\left(0,\frac{\pi}{2}\right)$

$$r^2 = 4\cos\theta$$



5) Graph the Curve $r^2 = 4\sin\theta$

Solution:

●The curve is symmetric about the x-axis because (r,θ) on the graph then

$(-r)^2 = 4\sin(\pi-\theta) \rightarrow r^2 = 4\sin\pi\cos\theta - 4\sin\theta\cos\pi \rightarrow r^2 = 4\sin\theta$, So

$(-r,\pi-\theta)$ on the graph

●The curve is symmetric about the y-axis because (r,θ) on the graph then

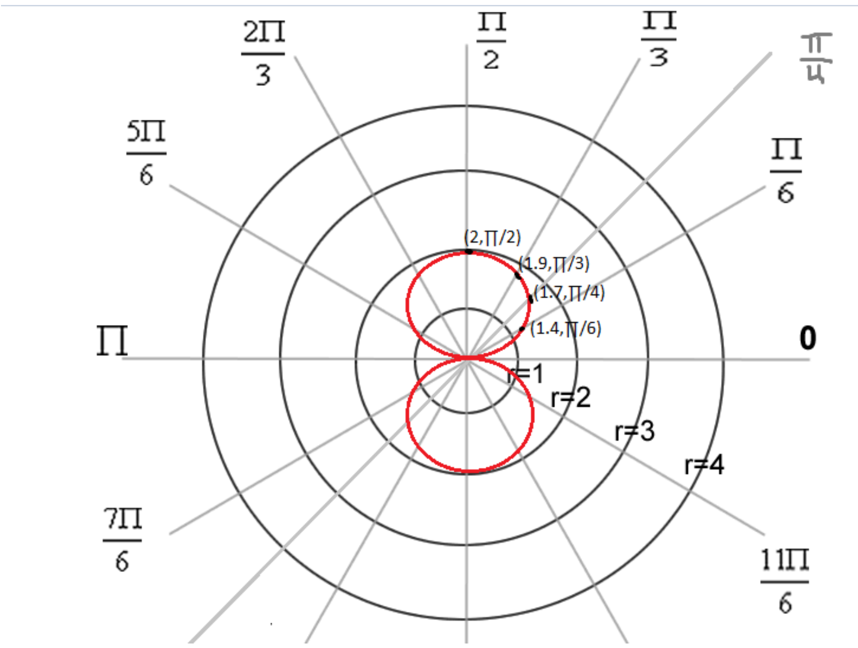
$r^2 = 4\sin(\pi-\theta) \rightarrow r^2 = 4\sin\pi\cos\theta - 4\sin\theta\cos\pi \rightarrow r^2 = 4\sin\theta$ So

$(r,\pi-\theta)$ on the graph

Together, these two symmetries imply symmetry about the origin point

θ	r^2	r	(r,θ)
0	0	0	(0,0)
$\frac{\pi}{6}$	$\frac{4}{2}$	±1.4	$\left(\pm1.4,\frac{\pi}{6}\right)$
$\frac{\pi}{4}$	$\frac{4}{\sqrt{2}}$	±1.7	$\left(\pm1.7,\frac{\pi}{4}\right)$
$\frac{\pi}{3}$	$\frac{4\sqrt{3}}{2}$	±1.9	$\left(\pm1.9,\frac{\pi}{3}\right)$
$\frac{\pi}{2}$	4	±2	$\left(\pm2,\frac{\pi}{2}\right)$

$r^2 = 4\sin\theta$



6) Graph the Curve $r = 2 + 2\cos\theta$

Solution:

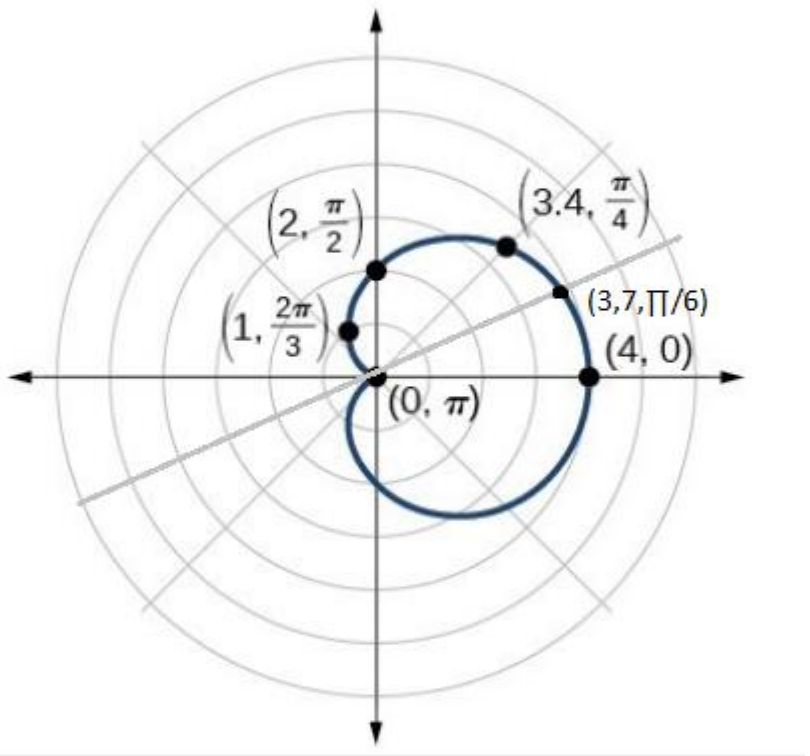
●The curve is symmetric about the x-axis because (r, θ) on the graph then

$$r = 2 + 2\cos(-\theta) \rightarrow r = 2 + 2\cos\theta \rightarrow (r, -\theta) \text{ on the graph}$$

There is not symmetric about the y-axis and the origin point

θ	r	(r, θ)
0	4	$(4,0)$
$\frac{\pi}{6}$	$2+\frac{2\sqrt{3}}{2}$	$\left(3.7, \frac{\pi}{6}\right)$
$\frac{\pi}{4}$	$2+\frac{2}{\sqrt{2}}$	$\left(3.4, \frac{\pi}{4}\right)$
$\frac{\pi}{3}$	$2+\frac{2}{2}$	$\left(3, \frac{\pi}{3}\right)$
$\frac{\pi}{2}$	2+0	$\left(2, \frac{\pi}{2}\right)$
$\frac{2\pi}{3}$	$2-\frac{2}{2}$	$\left(1, \frac{2\pi}{3}\right)$
π	2-2	$(0, \pi)$

$$r = 2 + 2\cos\theta$$



6) Graph the Curve $r = 2 + 2 \sin\theta$

Solution:

●The curve is symmetric about the y-axis because (r,θ) on the graph then

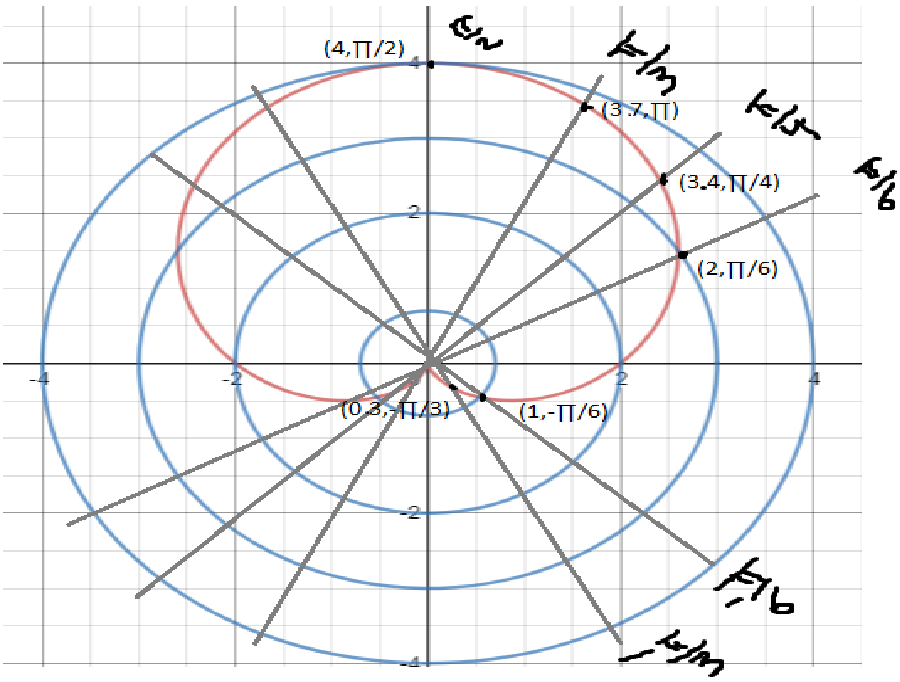
$r = 2 + 2 \sin(\pi - \theta) \rightarrow r = 2 + 2\sin\pi\cos\theta - 2\sin\theta\cos\pi \rightarrow$

$r = 2 + 2 \sin\theta$ So $(r, \pi - \theta)$ on the graph

There is not symmetric about the x-axis and the origin point

θ	r	(r,θ)
0	2	(2,0)
$\frac{\pi}{6}$	2+$\frac{2}{2}$	$\left(3,\frac{\pi}{6}\right)$
$\frac{-\pi}{6}$	2-$\frac{2}{2}$	$\left(1,\frac{\pi}{6}\right)$
$\frac{\pi}{4}$	2+$\frac{2}{\sqrt{2}}$	$\left(3.4,\frac{\pi}{4}\right)$
$\frac{\pi}{3}$	2+$\frac{2\sqrt{3}}{2}$	$\left(3.7,\frac{\pi}{3}\right)$
$-\frac{\pi}{3}$	2-$\frac{2\sqrt{3}}{2}$	$\left(0.3,-\frac{\pi}{3}\right)$
$\frac{\pi}{2}$	2+2	$\left(4,\frac{\pi}{2}\right)$

$r = 2 + 2 \sin\theta$



6) Graph the Curve $r = 4 + 3\ \sin\theta$

Solution:

●The curve is symmetric about the y-axis because (r, θ) on the graph then

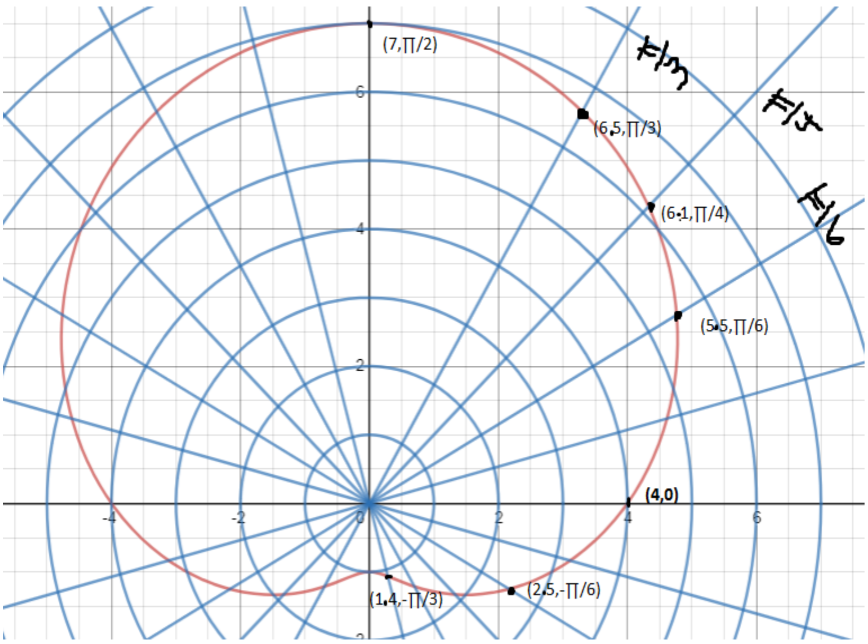
$r = 4 + 3\ \sin(\pi - \theta) \rightarrow r = 4 + 3\sin\pi\cos\theta - 3\sin\theta\cos\pi \rightarrow$

$r = 4 + 3\ \sin\theta$ So $(r, \pi - \theta)$ on the graph

There is not symmetric about the x-axis and the origin point

θ	r	(r,θ)
0	4	$(4,0)$
$\frac{\pi}{6}$	4+$\frac{3}{2}$	$\left(5.5, \frac{\pi}{6}\right)$
$\frac{-\pi}{6}$	4-$\frac{3}{2}$	$\left(2.5, \frac{\pi}{6}\right)$
$\frac{\pi}{4}$	4+$\frac{3}{\sqrt{2}}$	$\left(6.12, \frac{\pi}{4}\right)$
$\frac{\pi}{3}$	4+$\frac{3\sqrt{3}}{2}$	$\left(6.5, \frac{\pi}{3}\right)$
$-\frac{\pi}{3}$	4-$\frac{3\sqrt{3}}{2}$	$\left(1.4, -\frac{\pi}{3}\right)$
$\frac{\pi}{2}$	4+3	$\left(7, \frac{\pi}{2}\right)$

$r = 4 + 3\ \sin\theta$



7) Graph the Curve $r = 2 + 5\cos\theta$

Solution:

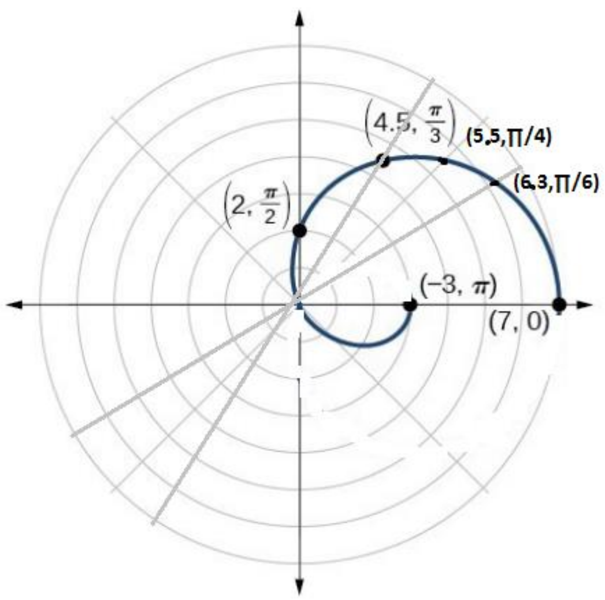
●The curve is symmetric about the x-axis because (r, θ) on the graph then

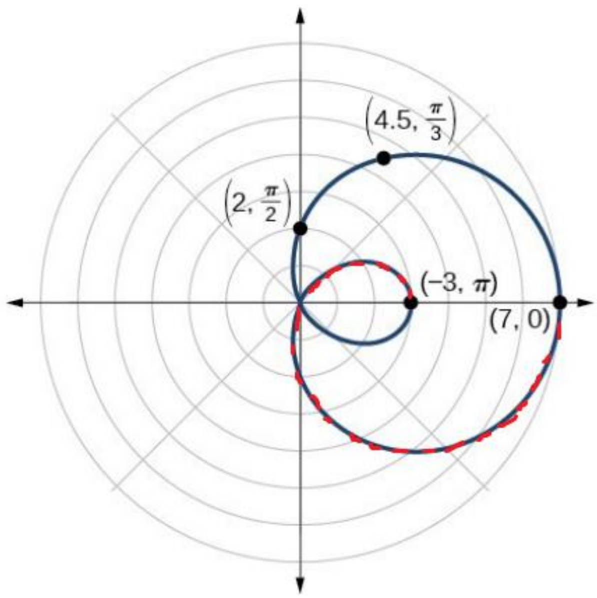
$$r = 2 + 5\cos(-\theta) \rightarrow r = 2 + 5\cos\theta \rightarrow (r, -\theta) \text{ on the graph}$$

There is not symmetric about the y-axis and the origin point

θ	r	(r, θ)
0	7	$(7,0)$
$\frac{\pi}{6}$	$2+\frac{5\sqrt{3}}{2}$	$\left(6.3, \frac{\pi}{6}\right)$
$\frac{\pi}{4}$	$2+\frac{5}{\sqrt{2}}$	$\left(5.5, \frac{\pi}{4}\right)$
$\frac{\pi}{3}$	$2+\frac{5}{2}$	$\left(4.5, \frac{\pi}{3}\right)$
$\frac{\pi}{2}$	2+0	$\left(2, \frac{\pi}{2}\right)$
$\frac{2\pi}{3}$	$2-\frac{5}{2}$	$\left(0.5, \frac{2\pi}{3}\right)$
π	2-5	$(-3, \pi)$

$$r = 2 + 5\cos\theta$$





because The curve is symmetric about the x-axis

5) Graph the Curve $r = \cos 2\theta$

Solution:

●The curve is symmetric about the *x*-axis because (r, θ) on the graph then

$r = \cos(-2\theta) \rightarrow r = \cos 2\theta$, So $(r, -\theta)$ on the graph

●The curve is symmetric about the *y*-axis because (r, θ) on the graph then

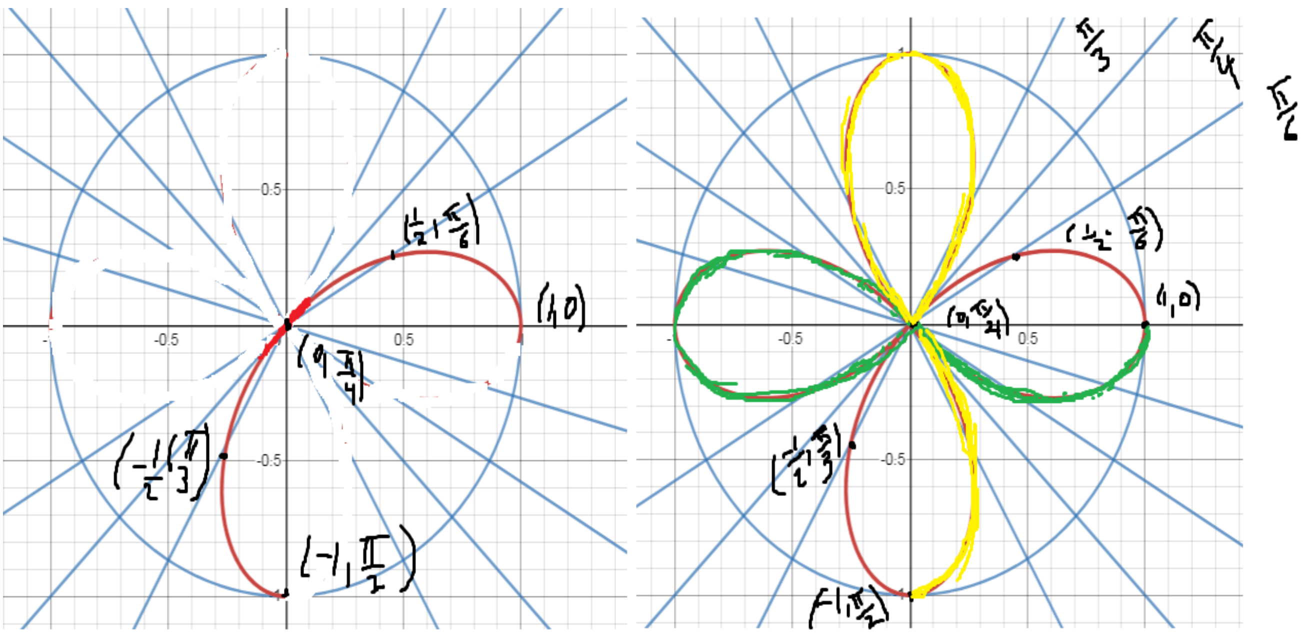
$r = \cos 2(\pi - \theta) \rightarrow r = \cos 2\pi \cos 2\theta + \sin 2\pi \sin 2\theta \rightarrow r = \cos 2\theta$, So

$(r, \pi - \theta)$ on the graph

Together, these two symmetries imply symmetry about the origin point

θ	r	(r, θ)
0	1	$(1,0)$
$\frac{\pi}{6}$	$\frac{1}{2}$	$\left(0.5, \frac{\pi}{6}\right)$
$\frac{\pi}{4}$	0	$\left(0, \frac{\pi}{4}\right)$
$\frac{\pi}{3}$	$\frac{-1}{2}$	$\left(-0.5, \frac{\pi}{3}\right)$
$\frac{\pi}{2}$	-1	$\left(-1, \frac{\pi}{2}\right)$

$r = \cos 2\theta$



because the curve is symmetric about the *x*-axis *and* the y-axis

5) Graph the Curve $r = \sin 2\theta$

Solution:

●The curve is symmetric about the x-axis because (r, θ) on the graph then

$-r = \sin 2(\pi - \theta) \rightarrow -\mathrm{r} = \sin 2\pi \cos 2\theta - \sin 2\theta \cos 2\pi \rightarrow -r = -\sin 2\theta$, $r = \sin 2\theta$ So $(r, \pi - \theta)$ on the graph

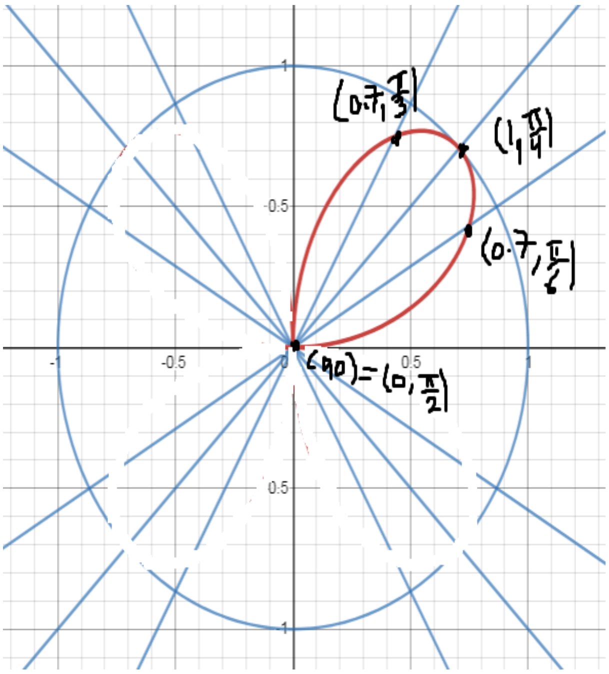
●The curve is symmetric about the y-axis because (r, θ) on the graph then

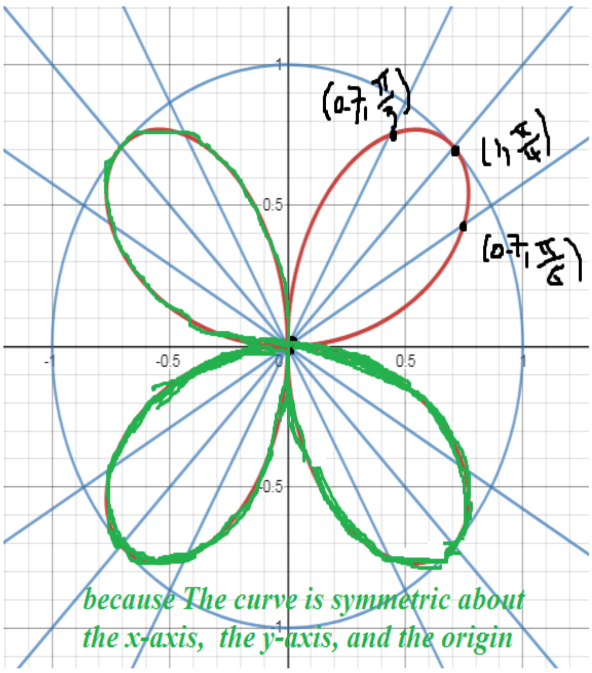
$-r = \sin -2\theta \rightarrow -r = -\sin 2\theta \rightarrow r = \sin 2\theta$, So $(-r, -\theta)$ on the graph

Together, these two symmetries imply symmetry about the origin point

θ	r	(r, θ)
0	0	$(0,0)$
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\left(0.8, \frac{\pi}{6}\right)$
$\frac{\pi}{4}$	1	$\left(1, \frac{\pi}{4}\right)$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\left(0.8, \frac{\pi}{3}\right)$
$\frac{\pi}{2}$	0	$\left(0, \frac{\pi}{2}\right)$

$r = \sin 2\theta$





5) Graph the Curve $r = \sin 3\theta$

Solution:

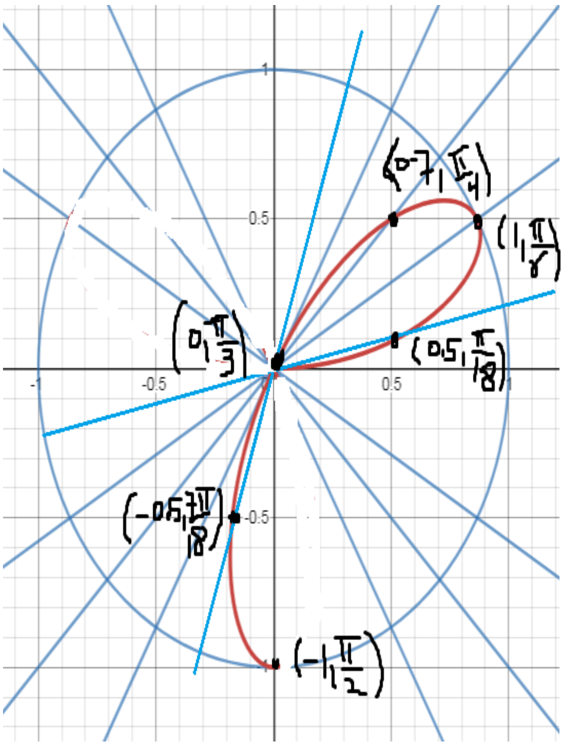
●The curve is symmetric about the y-axis because (r, θ) on the graph then

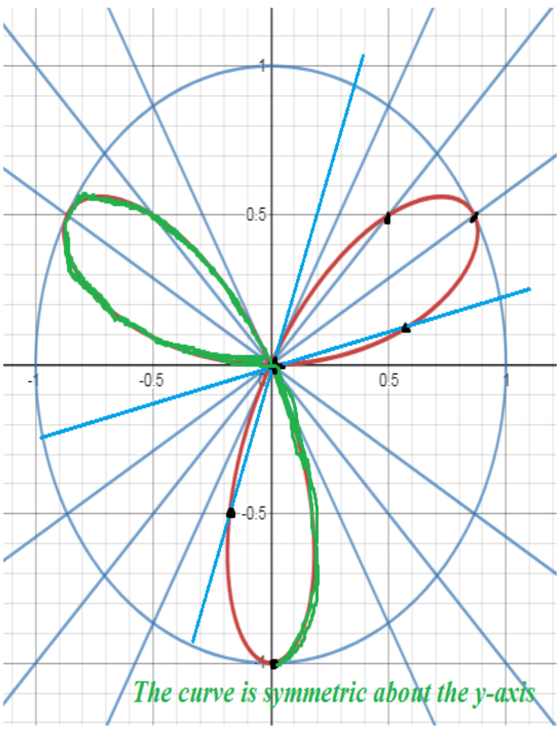
$-r = \sin(-3\theta) \rightarrow -r = -\sin 3\theta \rightarrow r = \sin 3\theta$ So $(-r, -\theta)$ on the graph

There is not symmetric about the x-axis and the origin point

θ	r	(r, θ)
0	0	$(0,0)$
$\frac{\pi}{18}$	$\frac{1}{2}$	$\left(0.5, \frac{\pi}{18}\right)$
$\frac{\pi}{6}$	1	$\left(1, \frac{\pi}{6}\right)$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\left(0.7, \frac{\pi}{4}\right)$
$\frac{\pi}{3}$	0	$\left(0, \frac{\pi}{3}\right)$
$\frac{7\pi}{18}$	$\frac{-1}{2}$	$\left(-0.5, \frac{7\pi}{18}\right)$
$\frac{\pi}{2}$	-1	$\left(-1, \frac{\pi}{2}\right)$

$r = \sin 3\theta$





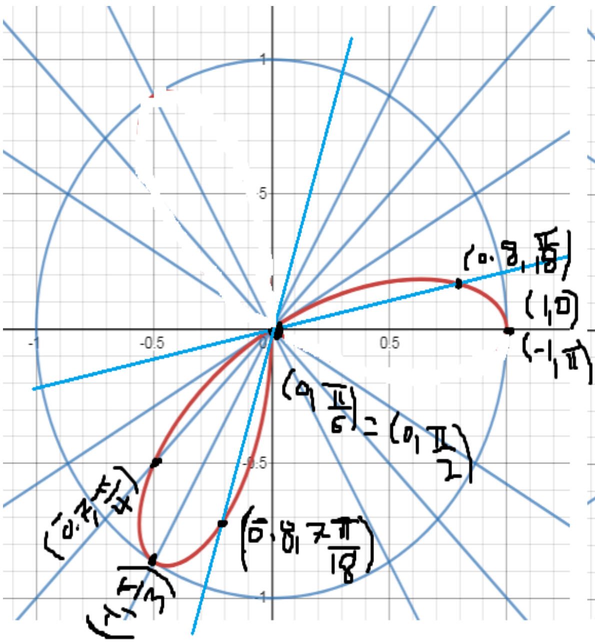
3) Graph the Curve $r = \cos 3\theta$

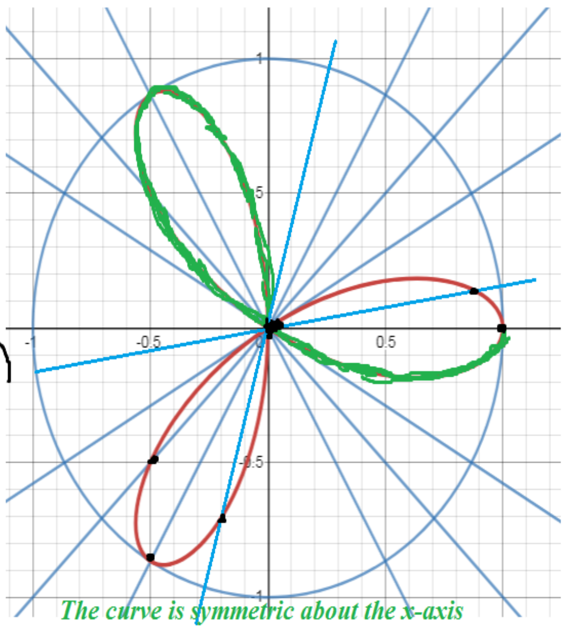
Solution: The curve is symmetric about the x-axis because (r, θ) on the graph then $r = \cos(-3\theta) \to r = \cos 3\theta \to (r, -\theta)$ on the graph

There is not symmetric about the y-axis and the origin point

θ	r	(r, θ)
0	1	$(1,0)$
$\frac{\pi}{18}$	$\frac{\sqrt{3}}{2}$	$\left(0.8, \frac{\pi}{18}\right)$
$\frac{\pi}{6}$	0	$\left(0, \frac{\pi}{6}\right)$
$\frac{\pi}{4}$	$\frac{-1}{\sqrt{2}}$	$\left(-0.7, \frac{\pi}{4}\right)$
$\frac{\pi}{3}$	-1	$\left(-1, \frac{\pi}{3}\right)$
$\frac{7\pi}{18}$	$\frac{-\sqrt{3}}{2}$	$\left(-0.8, \frac{\pi}{18}\right)$
$\frac{\pi}{2}$	0	$\left(0, \frac{\pi}{2}\right)$
$\frac{2\pi}{3}$	-1	$\left(-1, \frac{2\pi}{3}\right)$
π	-1	$(-1, \pi)$

$r = \cos 3\theta$





5) Graph the Curve $r^2 = 4\cos 2\theta$

Solution:

●The curve is symmetric about the *x*-axis because (r,θ) on the graph then

$r^2 = 4\cos(-2\theta) \rightarrow r^2 = 4\cos 2\theta$, So $(r,-\theta)$ on the graph

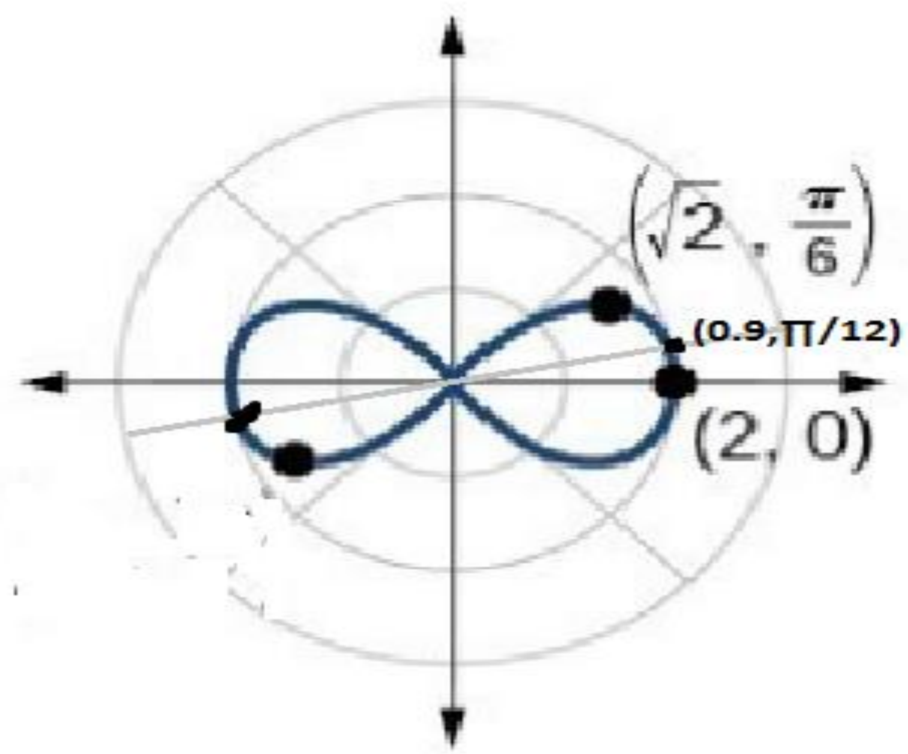
●The curve is symmetric about the *y*-axis because (r,θ) on the graph then

$\boldsymbol{r^2 = 4\cos 2(\pi-\theta) \rightarrow r^2 = 4\cos 2\pi \cos 2\theta + 4\sin 2\pi \sin 2\theta \rightarrow r^2 = 4\cos 2\theta}$, So $(r,\pi-\theta)$ on the graph

Together, these two symmetries imply symmetry about the origin point

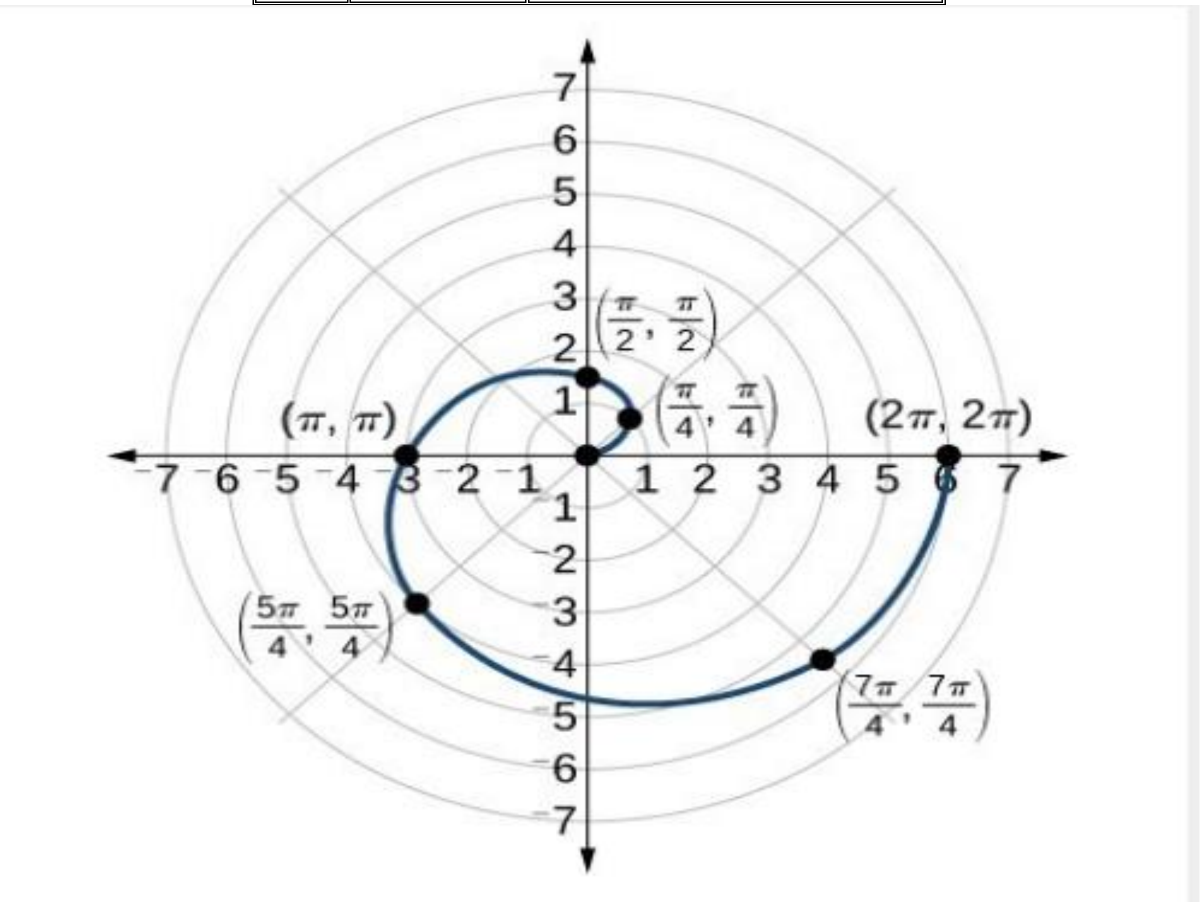
θ	$\boldsymbol{r^2}$	r	(r,θ)
0	4	± 2	$(\pm 2,0)$
$\frac{\pi}{12}$	$\frac{\sqrt{3}}{2}$	$\frac{\pm\sqrt[4]{3}}{\sqrt{2}}$	$\left(\pm 0.9,\frac{\pi}{12}\right)$
$\frac{\pi}{6}$	$\frac{4}{2}$	$\pm\sqrt{2}$	$\left(\pm 1.4,\frac{\pi}{6}\right)$
$\frac{\pi}{4}$	0	0	$\left(0,\frac{\pi}{4}\right)$

$r^2 = 4\cos 2\theta$



5) Graph the Curve $r = \theta$ where $0 \le \theta \le 2\pi$

θ	r	(r, θ)
0	0	$(0,0)$
$\frac{\pi}{6}$	0.52	$\left(0.52, \frac{\pi}{6}\right)$
$\frac{\pi}{4}$	0.78	$\left(0.78, \frac{\pi}{4}\right)$
$\frac{\pi}{3}$	1.04	$\left(1.04, \frac{\pi}{3}\right)$
$\frac{\pi}{2}$	1.57	$\left(1.57, \frac{\pi}{2}\right)$
$\frac{2\pi}{3}$	2.09	$\left(2.09, \frac{2\pi}{3}\right)$
π	3.14	$(3.14, \pi)$
$\frac{3\pi}{2}$	4.71	$\left(4.71, \frac{3\pi}{2}\right)$
$2\,\pi$	6.28	$(6.28, 2\,\pi)$



Areas and Lengths in Polar Coordinates

This section shows how to calculate areas of plane regions, lengths of curves, and areas of surfaces of revolution in polar coordinates.

Area in the Plane

The region OTS in Figure 10.48 is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and the curve $r = f(\theta)$. We approximate the region with n nonoverlapping fan-shaped circular sectors based on a partition P of angle TOS. The typical sector has radius $r_k = f(\theta_k)$ and central angle of radian measure $\Delta\theta_k$. Its area is $\Delta\theta_k/2\pi$ times the area of a circle of radius r_k, or

$$A_k = \frac{1}{2} r_k^2 \, \Delta\theta_k = \frac{1}{2} \left(f(\theta_k)\right)^2 \Delta\theta_k \, .$$

The area of region OTS is approximately

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n} \frac{1}{2} \left(f(\theta_k)\right)^2 \Delta\theta_k \, .$$

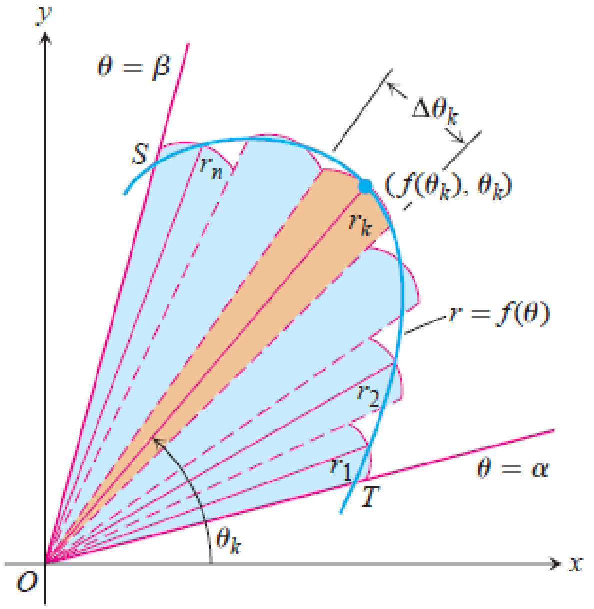


FIGURE 10.48 To derive a formula for the area of region *OTS*, we approximate the region with fan-shaped circular sectors.

If f is continuous, we expect the approximations to improve as the norm of the partition $\|P\| \to 0$, and we are led to the following formula for the region's area:

$$A = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \frac{1}{2}\left(f(\theta_k)\right)^2 \Delta\theta_k$$
$$= \int_{\alpha}^{\beta} \frac{1}{2}\left(f(\theta)\right)^2 d\theta.$$

Area of the Fan-Shaped Region Between the Origin and the Curve $r = f(\theta)$, $\alpha \le \theta \le \beta$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta.$$

This is the integral of the **area differential** (Figure 10.49)

$$dA = \frac{1}{2} r^2 \, d\theta = \frac{1}{2}(f(\theta))^2 \, d\theta.$$

EXAMPLE 1 Finding Area

Find the area of the region in the plane enclosed by the cardioid $r = 2(1 + \cos\theta)$.

$$
\begin{aligned}
\int_{\theta=0}^{\theta=2\pi} \frac{1}{2} r^2 \, d\theta &= \int_0^{2\pi} \frac{1}{2} \cdot 4(1 + \cos\theta)^2 \, d\theta \\
&= \int_0^{2\pi} 2(1 + 2\cos\theta + \cos^2\theta) \, d\theta \\
&= \int_0^{2\pi} \left(2 + 4\cos\theta + 2\,\frac{1 + \cos 2\theta}{2}\right) d\theta \\
&= \int_0^{2\pi} (3 + 4\cos\theta + \cos 2\theta) \, d\theta \\
&= \left[3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} = 6\pi - 0 = 6\pi. \qquad \blacksquare
\end{aligned}
$$

Area of the Region $0 \le r_1(\theta) \le r \le r_2(\theta), \qquad \alpha \le \theta \le \beta$

$$A = \int_\alpha^\beta \frac{1}{2} r_2^2 \, d\theta - \int_\alpha^\beta \frac{1}{2} r_1^2 \, d\theta = \int_\alpha^\beta \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta \qquad (1)$$

EXAMPLE 3 Finding Area Between Polar Curves

Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.

Solution We sketch the region to determine its boundaries and find the limits of integration (Figure 10.53). The outer curve is $r_2 = 1$, the inner curve is $r_1 = 1 - \cos\theta$, and θ runs from $-\pi/2$ to $\pi/2$. The area, from Equation (1), is

$$\begin{aligned}
A &= \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta \\
&= 2\int_0^{\pi/2} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta \qquad \text{Symmetry} \\
&= \int_0^{\pi/2} (1 - (1 - 2\cos\theta + \cos^2\theta))\, d\theta \\
&= \int_0^{\pi/2} (2\cos\theta - \cos^2\theta)\, d\theta = \int_0^{\pi/2}\left(2\cos\theta - \frac{1 + \cos 2\theta}{2}\right) d\theta \\
&= \left[2\sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/2} = 2 - \frac{\pi}{4}.
\end{aligned}$$

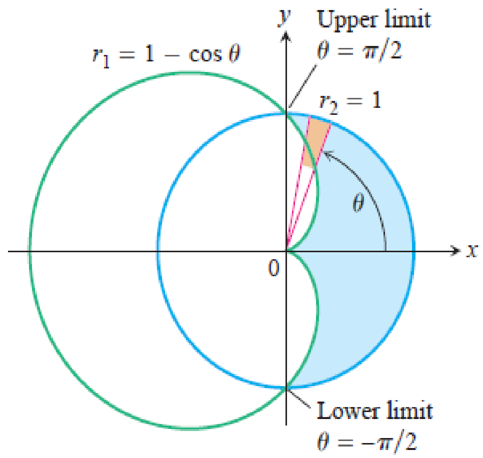


FIGURE 10.53 The region and limits of integration in Example 3.

Length of a Polar Curve

We can obtain a polar coordinate formula for the length of a curve $r = f(\theta)$, $\alpha \leq \theta \leq \beta$, by parametrizing the curve as

$$x = r\cos\theta = f(\theta)\cos\theta, \qquad y = r\sin\theta = f(\theta)\sin\theta, \qquad \alpha \leq \theta \leq \beta. \tag{2}$$

The parametric length formula, Equation (1) from Section 6.3, then gives the length as

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta.$$

This equation becomes

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

when Equations (2) are substituted for x and y (Exercise 33).

Length of a Polar Curve

If $r = f(\theta)$ has a continuous first derivative for $\alpha \leq \theta \leq \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the length of the curve is

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta. \tag{3}$$

EXAMPLE 4 Finding the Length of a Cardioid

Find the length of the cardioid $r = 1 - \cos\theta$.

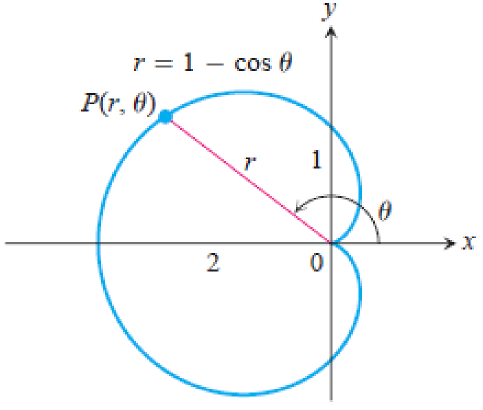


FIGURE 10.54 Calculating the length of a cardioid (Example 4).

Solution We sketch the cardioid to determine the limits of integration (Figure 10.54). The point $P(r, \theta)$ traces the curve once, counterclockwise as θ runs from 0 to 2π, so these are the values we take for α and β.

With

$$r = 1 - \cos\theta, \qquad \frac{dr}{d\theta} = \sin\theta,$$

we have

$$\begin{aligned} r^2 + \left(\frac{dr}{d\theta}\right)^2 &= (1 - \cos\theta)^2 + (\sin\theta)^2 \\ &= 1 - 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{1} = 2 - 2\cos\theta \end{aligned}$$

and

$$\begin{aligned} L &= \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta \\ &= \int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta \qquad 1 - \cos\theta = 2\sin^2\frac{\theta}{2} \\ &= \int_0^{2\pi} 2\left|\sin\frac{\theta}{2}\right| d\theta \\ &= \int_0^{2\pi} 2\sin\frac{\theta}{2}\, d\theta \qquad \sin\frac{\theta}{2} \ge 0 \quad \text{for} \quad 0 \le \theta \le 2\pi \\ &= \left[-4\cos\frac{\theta}{2}\right]_0^{2\pi} = 4 + 4 = 8. \end{aligned}$$

MATHEMATICS DEPARTMENT COLLEGE OF EDUCATION FOR PURE SCIENCES

UNIVERSITY OF ANBAR

Lecture Note

Advance Calculus

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Acknowledgements

[illegible]

Contents

Chapter 1

Introduction: Infinite Sequences

The term sequence in mathematics is used to describe an unending succession of numbers. The numbers in a sequence are called the terms of the sequence. For example

$$< 1, 3, 5, \dots >$$
$$< 1, \frac{1}{2}, \frac{1}{4}, \dots >$$

Definition
An infinite **sequence** is a function whose domain is the set of positive integers.
Let $U_n =< U_1, U_2, U_3, \dots >$ be infinite sequence and by definition above $U_n : \mathbb{N} \longrightarrow \mathbb{R}$ (U_n is called the n^{th} term of the sequence). For example

$$U_n = 2, 4, 6, \dots, 2n, \dots$$
$$U_n = 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, 1 - \frac{1}{n}, \dots$$
$$U_n = 1, -1, 1, \dots, (-1)^n + 1, \dots$$

Examples Find a formula for n^{th} term of the infinite sequence?

$$< 1, -4, 9, -16, \dots >$$
$$< 3, 8, 15, \dots >$$
$$< 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \dots >$$
$$< 1, \frac{1}{2}, \frac{1}{4}, \dots >$$

Solution :

$$< 1, -4, 9, -16, \dots, (-1)^{n+1}n^2, \dots >$$
$$< 3, 8, 15, \dots, n^2 - 1, \dots >$$
$$< 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \dots, \frac{1}{n!} >$$
$$< 1, \frac{1}{2}, \frac{1}{4}, \dots, >$$

1.1 Graphs of Sequences

The graph of the sequence U_n is the graph of the equation

$$f(n) = U_n, n = 1, 2, 3, \dots$$

For example the graph of sequence $U_n = 1, \frac{1}{2}, \frac{1}{3}, \dots$ We need to find the formula of sequence $U_n =$

Definition

A sequence U_n is said to **converge** and have limit $L \in \mathbb{R}$ if given any $\epsilon > 0$, there is a positive integer N such that for all $n > N$,

$$|U_n - L| < \epsilon.$$

In this case, we write $\lim_{n \longrightarrow \infty} U_n = L$.
Note that

$$\text{if } \lim_{n \longrightarrow \infty} U_n = L, \text{ then } U_n \text{ converges.}$$

$$\text{if } \lim_{n \longrightarrow \infty} U_n \Longrightarrow \infty, \text{ then } U_n \text{ diverges.}$$

Examples Do these sequences converge or diverge?

$$U_n = \frac{1 - 2n}{1 + 2n}.$$

$$U_n = \frac{n^2 - 2n + 1}{n - 1}.$$
$$U_n = \frac{ln(n)}{n}.$$
$$U_n = sin(\frac{\Pi}{2} + \frac{1}{n}).$$
$$U_n = \left(1 - \frac{1}{n}\right)^n.$$

Solution :

$$\lim_{n\longrightarrow\infty} \frac{1-2n}{1+2n} = \lim_{n\longrightarrow\infty} \frac{1/n - 2}{1/n + 2} = \lim_{n\longrightarrow\infty} \frac{-2}{2} = -1, \text{ thus it converges.}$$

$$\lim_{n\longrightarrow\infty} \frac{n^2 - 2n + 1}{n - 1} = \lim_{n\longrightarrow\infty} \frac{(n-1)(n-1)}{n-1} = \lim_{n\longrightarrow\infty} n-1 = \infty, \text{ thus it diverges.}$$

$$\lim_{n\longrightarrow\infty} \frac{ln(n)}{n} = \lim_{n\longrightarrow\infty} \frac{1/n}{1}, \text{ (by using L'Hopital's rule) } \Longrightarrow \lim_{n\longrightarrow\infty} \frac{1}{n} = 0, \text{ thus it converges.}$$

$$\lim_{n\longrightarrow\infty} sin(\frac{\Pi}{2}+\frac{1}{n}) = sin(\lim_{n\longrightarrow\infty} \frac{\Pi}{2} + \lim_{n\longrightarrow\infty} \frac{1}{n}) = sin(\frac{\Pi}{2}) = 1, \text{ thus it converges.}$$

$$\lim_{n\longrightarrow\infty} \left(1 - \frac{1}{n}\right)^n$$

the solution will be in the lecturer.

THEOREM: The Sandwich Theorem for Sequences
Let$< a_n >$ and $< b_n >$ be sequences of real numbers. If $a_n \leq b_n \leq c_n$ holds for all n beyond some index N, and if $\lim_{n\longrightarrow\infty} a_n = \lim_{n\longrightarrow\infty} c_n = L$, then $\lim_{n\longrightarrow\infty} b_n = L$ also.
Applying the sandwich theorem.

Example Is the sequence $U_n = \dfrac{\cos n}{n}$ converge or diverge?

Solution:

$$\begin{aligned}
\cos n \Rightarrow & -1 \leq \cos n \leq 1 \\
\Rightarrow & \frac{-1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n} \\
\Rightarrow & 0 \leq \frac{\cos n}{n} \leq 0, \text{ since } \lim_{n\longrightarrow\infty} \frac{-1}{n} = \lim_{n\longrightarrow\infty} \frac{1}{n} = 0.
\end{aligned}$$

Thus, the sequence $\dfrac{\cos n}{n} \Rightarrow 0$ converges.
Note that a sequence is called increasing if $u_n < u_{n+1}$ for all n as $< \frac{2^n-1}{2^n} >$. **Similarly, a sequence is decreasing if** $u_n > u_{n+1}$ for all n as $< \frac{n+1}{n} >$.

THEOREM
The following sequences converge to the limits listed below:

- $\lim_{n\longrightarrow\infty} \frac{\ln n}{n} = 0$
- $\lim_{n\longrightarrow\infty} \sqrt[n]{n} = 1$
- $\lim_{n\longrightarrow\infty} x^{\frac{1}{n}} = 1, (x > 0)$
- $\lim_{n\longrightarrow\infty} x^n = 0, (|x| < 1)$
- $\lim_{n\longrightarrow\infty} \left(1 + \frac{x}{n}\right)^n = e^x$, (for any x).
-

Exercises

Q1) Given a formula for the n^{th} term U_n of a sequence $< u_n >$. Find the values of u_1, u_2, u_3, u_4, and u_5.

- $U_n = \frac{1-n}{n^2}$
- $U_n = \frac{2^n}{2^{n+1}}$
- $U_n = \frac{1}{n!}$

Q2) Find the sequence

- If $u_n = u_{n-1} + 1$, where $u_1 = 1$.
- If $u_{n+1} = u_n + \frac{1}{2^n}$, where $u_1 = 2$.
- If $u_{n+1} = u_n + u_{n-1}$, where $u_1 = 1, u_2 = 1$.

Q3) Find a formula for the n^{th} term of the sequence.

- $< -3, -2, -1, 0, 1, \ldots >$
- $< 1, 5, 9, 13, 17, \ldots >$
- $< 1, 0, 1, 0, 1, \ldots >$

Q4) which of the sequence below converge, and which diverge?

- $U_n = \dfrac{1-5n^4}{n^4+8n^3}$.
- $U_n = \ln\left(\dfrac{n+1}{n}\right)^n$.
- $U_n = \dfrac{\ln n}{n^{\frac{1}{n}}}$.
- $U_n = \left(n - \sqrt{n^2-n}\right)$.
- $U_n = \dfrac{\sin n}{n}$.

Chapter 2

Introduction: Infinite Series

Infinite series is an expression that can be written in the form

$$\sum_{n=1}^{\infty} U_n = u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

Some Special Series

1)Geometric series : If $a, r \in \mathbb{R}$, then a series of the form

$$\sum_{n=1}^{\infty} ar^n = a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$$

for example

$$\sum_{n=1}^{\infty} 3^n = 3 + 9 + 27 + \cdots$$

2)P− series : If $p \in \mathbb{R}$, then a series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called P-series. For example

$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

3)Harmonic series : It is a special case of p-series when $p = 1$. Therefore, we have form

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

4)Telescoping series : The series has a form

$$\sum_{n=1}^{\infty} \Big(U_n - U_{n+1} \Big),$$

for example

$$(1-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{4})+\ldots+(\frac{1}{n}-\frac{1}{n+1})+\cdots$$

2.1 Convergence and Divergence Test for Infinite Positive Series

Test1 : Geometric series

$$\sum_{n=1}^{\infty} ar^n = a + ar + ar^2 + \ldots + ar^{n-1} + \ldots$$

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1}$$

$$rS_n = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n$$

$$S_n = \frac{a(1-r^n)}{1-r},$$

where a first term and r common ratio ($r = \frac{U_2}{U_1} = \frac{U_3}{U_2} = \ldots$)

Test condition

1) If $|r| < 1$, then $r^n \longrightarrow 0$ as $n \longrightarrow \infty$ and the series converges with sum total $\frac{a}{1-r}$.
2) If $|r| \geq 1$, then $r^n \longrightarrow \infty$ as $n \longrightarrow \infty$ and the series diverges.

Summary If $|r| < 1$, then U_n converges. If $|r| \geq 1$, then U_n diverges.
Example:

$$\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\ldots$$

$r = \frac{1/27}{1/9} = \frac{1/81}{1/27} = 1/3 < 1$ so the series converges with sum $\frac{1/9}{1-1/3} = 1/6$.
Example: Dose

$$2+4+8+16+\ldots$$

converge or diverge?

$r = 4/2 = 8/4 = 2 \geq 1$ so the series diverges.

Test2 : Integral test
1)$f(n) = f(x)$ able to integral.
2)$f(x)$ is decreasing function.
3)If $\lim_{L\longrightarrow\infty}\int_a^L f(x)dx$ is exist then the series converges otherwise diverges.

Example: Dose

$$\frac{1}{e}+\frac{2}{e^4}+\frac{3}{e^9}+\ldots=\sum_{n=1}^{\infty}\frac{n}{e^{n^2}}$$

converge or diverge?

1)$f(n) = f(x) = \frac{x}{e^{x^2}} = xe^{-x^2}$
2)$f(x)$ is decreasing function because $f^{'}(x) = -2x^2e^{-x^2} + e^{-x^2}$.
3)

$$\lim_{L\longrightarrow\infty}\int_a^L f(x)dx = \lim_{L\longrightarrow\infty}\int_1^L xe^{-x^2}dx$$

$$\frac{-1}{2}\lim_{L\longrightarrow\infty}\left(e^{-L^2}-e^{-1}\right)=\frac{1}{2e},$$

so the series converges.

Test3 : Ratio test
Let $\sum_{n=1}^{\infty}U_n$ be a series of positive terms and let $\rho = \lim_{n\longrightarrow\infty}\frac{U_{n+1}}{U_n}$
1)If $\rho < 1$, then $\sum_{n=1}^{\infty}U_n$ converges.
2)If $\rho > 1$ or $\rho \Longrightarrow \infty$, then $\sum_{n=1}^{\infty}U_n$ diverges.
3)If $\rho = 1$, then the ratio test fails.
Example: Dose $\sum_{n=1}^{\infty}\frac{1}{n!}$ converge or diverge?
Let $U_n = \frac{1}{n!}$, then $U_{n+1} = \frac{1}{(n+1)!}$. Consider

$$\rho = \lim_{n\longrightarrow\infty}\frac{U_{n+1}}{U_n}$$

$$= \lim_{n\longrightarrow\infty}\frac{1/(n+1)!}{1/n!}$$

$$= \lim_{n\longrightarrow\infty}\frac{n!}{(n+1)!}$$

$$= \lim_{n\longrightarrow\infty}\frac{n!}{(n+1)n!}$$

$$= \lim_{n\longrightarrow\infty}\frac{1}{n+1} = 0 < 1$$

thus, $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges.

Test4 : Root test

Let $\sum_{n=1}^{\infty} U_n$ be a series of positive terms and let $\rho = \lim_{n \longrightarrow \infty} \sqrt[n]{U_n}$

1)If $\rho < 1$, then $\sum_{n=1}^{\infty} U_n$ converges.

2)If $\rho > 1$ or $\rho \Longrightarrow \infty$, then $\sum_{n=1}^{\infty} U_n$ diverges.

3)If $\rho = 1$, then the ratio test fails.

Example: Does $\sum_{n=1}^{\infty} \frac{1}{n^n}$ converge or diverge?
We have $\sqrt[n]{U_n} = \sqrt[n]{\frac{1}{n^n}} = \frac{1}{n}$. Consider

$$\begin{aligned} \rho &= \lim_{n \longrightarrow \infty} \sqrt[n]{U_n} \\ &= \lim_{n \longrightarrow \infty} \frac{1}{n} \\ &= 0 < 1, \end{aligned}$$

thus, $\sum_{n=1}^{\infty} \frac{1}{n^n}$ converges.

Test5 : P − Test

Let $\sum_{n=1}^{\infty} \frac{1}{n^p}$ be a P-series. Then

1)If $P > 1$, then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

2)If $0 < P \leq 1$, then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

Example: Does $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$ converge or diverge?
It is clearly $p = 1/5$ and $0 < \frac{1}{5} \leq 1$, then $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$ diverges

Test 6 : Comparison Ratio Test

Let $\sum_{n=1}^{\infty} U_n$ be a series of positive terms. Further,

let $\sum_{n=1}^{\infty} C_n$ be a known convergent series of positive terms, and

let $\sum_{n=1}^{\infty} d_n$ be a known divergent series of positive terms

1)If

$$\lim_{n\longrightarrow\infty}\frac{U_n}{C_n}=L,\ \text{then}\ \sum_{n=1}^{\infty}U_n\ \text{converges.}$$

2)If

$$\lim_{n\longrightarrow\infty}\frac{U_n}{d_n}=L>0,\ \text{then}\ \sum_{n=1}^{\infty}U_n\ \text{diverges.}$$

3)If

$$\lim_{n\longrightarrow\infty}\frac{U_n}{d_n}\Longrightarrow\infty,\ \text{then}\ \sum_{n=1}^{\infty}U_n\ \text{diverges.}$$

Example: Does $\sum_{n=1}^{\infty}\frac{5^n+1}{3^n}$ converge or diverge?

We have $\frac{5^n}{3^n}$ which we know is divergent by geometric test $r=5/3>1$. So, let $d_n=\frac{5^n}{3^n}$. Now

$$\begin{aligned}\lim_{n\longrightarrow\infty}\frac{U_n}{d_n}&=\lim_{n\longrightarrow\infty}\frac{5^n+1/3^n}{5^n/3^n}\\&=\lim_{n\longrightarrow\infty}\frac{5^n+1}{5^n}\\&=1>0.\end{aligned}$$

Thus by C.R.T (Comparison Ratio Test) part 2, the series $\sum_{n=1}^{\infty}\frac{5^n+1}{3^n}$ diverge.

2.2 Alternating series

If $<u_n>$ is a sequence of positive term. Then, $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ is called an alternating series as $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n!}$.

Absolutely convergent
A series $\sum_{n=1}^{\infty}u_n$ is said to be absolutely convergent if and only if $\sum_{n=1}^{\infty}|u_n|$ converges. For example $\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}$.

Conditionally convergent
If a series $\sum_{n=1}^{\infty}u_n$ satisfies each of the following three conditions, then $\sum_{n=1}^{\infty}u_n$ conditionally convergent

- $\sum_{n=1}^{\infty} u_n$ is an alternating series.
- $\lim_{n\longrightarrow\infty} u_n = 0$.
- $|u_{n+1}| \leq |u_n|$.

For example: Does $\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+1}$ converge or diverge?

Solution: We have

- $\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+1} = \frac{-1}{2} + \frac{2}{5} - \frac{3}{10} + \dots$ which is an alternating series.
- $\lim_{n\longrightarrow\infty}(-1)^n \frac{n}{n^2+1} = 0$.
- It is clearly $|u_{n+1}| \leq |u_n|$.

Thus, the series is a conditionally convergent.

Chapter 3

Power Series

The series $\sum_{n=0}^{\infty} a_n(x-b)^n$ is called a **power series** in $(x-b)$, where a_n is a sequence in $\mathbb{R}$. When $b=0$, we say that $\sum_{n=0}^{\infty} a_n x^n$ is a power series.

Example: For what values of x does the series $\sum_{n=0}^{\infty} \frac{(x-3)^n}{n!}$ converge?
Apply the ratio test

$$\begin{aligned}
\lim_{n\longrightarrow\infty} |\frac{U_{n+1}}{U_n}| < 1 &\Longrightarrow |\frac{(x-3)^{n+1} n!}{(n+1)!(x-3)^n}| \\
&\Longrightarrow \lim_{n\longrightarrow\infty} |\frac{(x-3)^n(x-3)n!}{(n+1)n!(x-3)^n}| \\
&\Longrightarrow \lim_{n\longrightarrow\infty} |\frac{(x-3)}{(n+1)}| \\
&\Longrightarrow |(x-3)| \lim_{n\longrightarrow\infty} \frac{1}{n+1} \\
&\Longrightarrow |(x-3)| \times 0 = 0 < 1, \text{ for all } x.
\end{aligned}$$

Thus, the series converge for all $x \in \mathbb{R}$.
Example: Find the series' interval of convergence for $\sum_{n=0}^{\infty} (\ln x)^n$?
Apply the ratio test $\lim_{n\longrightarrow\infty} |\frac{U_{n+1}}{U_n}| < 1$
Exercise

Example: Find the series' interval of convergence for $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x+2)^n}{n2^n}$?

Apply the ratio test $\lim_{n\longrightarrow\infty}|\frac{U_{n+1}}{U_n}| < 1$

$$\begin{aligned}
&\Longrightarrow |\frac{(x+2)^{n+1}n2^n}{(n+1)2^{n+1}(x+2)^n}| < 1\\
&\Longrightarrow \frac{|(x+2)|}{2}\lim_{n\longrightarrow\infty}\frac{n}{n+1} < 1\\
&\Longrightarrow \frac{|(x+2)|}{2} < 1\\
&\Longrightarrow -2 < x+2 < 2\\
&\Longrightarrow -4 < x < 0.
\end{aligned}$$

Now, when $x=-4$ we have $\sum_{n=1}^{\infty}\frac{-1}{n}$ divergent series; when $x=0$ we have $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$, the alternating series which converges conditionally.
Thus, the interval of convergence is $-4 < x \leq 0$.

3.1 Taylor series and Manclaurin series

Suppose f is a given function which is k times differentiable at a given point $x=a$. Then, **Taylor series** is

$$f(x)=f(a)+f^{'}(a)(x-a)+\frac{f^{''}(a)(x-a)^2}{2!}+\ldots+\frac{f^k(a)(x-a)^k}{k!}+\ldots=\sum_{n=0}^{\infty}\frac{f^n(a)(x-a)^n}{n!}.$$

When $a=0$, in this case, the Taylor series is called **Maclaurin series**, and is given by

$$f(x)=f(0)+f^{'}(0)x+\frac{f^{''}(0)x^2}{2!}+\ldots+\frac{f^k(0)x^k}{k!}+\ldots=\sum_{n=0}^{\infty}\frac{f^n(0)x^n}{n!}$$

Example: Find the Taylor series expansion of $f(x)=sin(x)$ about the point $a=\frac{\Pi}{2}$?

We have

$$
\begin{aligned}
f(x) = sin(x) &\Longrightarrow f(\frac{\Pi}{2}) = sin(\frac{\Pi}{2}) = 1,\\
f'(x) = cos(x) &\Longrightarrow f^{'}(\frac{\Pi}{2}) = cos(\frac{\Pi}{2}) = 0,\\
f''(x) = -sin(x) &\Longrightarrow f^{''}(\frac{\Pi}{2}) = -sin(\frac{\Pi}{2}) = -1,\\
f'''(x) = -cos(x) &\Longrightarrow f^{'''}(\frac{\Pi}{2}) = -cos(\frac{\Pi}{2}) = 0,\\
f^4(x) = sin(x) &\Longrightarrow f^4(\frac{\Pi}{2}) = sin(\frac{\Pi}{2}) = 1,
\end{aligned}
$$

and so on. Thus, the Taylor series is

$$
\begin{aligned}
f(x) =& f(\frac{\Pi}{2}) + f^{'}(\frac{\Pi}{2})(x - \frac{\Pi}{2}) + \frac{f^{''}(\frac{\Pi}{2})(x - \frac{\Pi}{2})^2}{2!} + \ldots + \frac{f^k(\frac{\Pi}{2})(x - \frac{\Pi}{2})^k}{k!} + \ldots\\
=& 1 - \frac{(x - \Pi/2)^2}{2!} + \frac{(x - \Pi/2)^4}{4!} + \ldots
\end{aligned}
$$

Example: Find the Manclaurin series expansion of $f(x) = sin(x)$?

We have

$$
\begin{aligned}
f(x) = sin(x) &\Longrightarrow f(0) = sin(0) = 0,\\
f'(x) = cos(x) &\Longrightarrow f^{'}(0) = cos(0) = 1,\\
f''(x) = -sin(x) &\Longrightarrow f^{''}(0) = -sin(0) = 0,\\
f'''(x) = -cos(x) &\Longrightarrow f^{'''}(0) = -cos(0) = -1,\\
f^4(x) = sin(x) &\Longrightarrow f^4(0) = sin(0) = 0,
\end{aligned}
$$

and so on. Thus, the Manclaurin series is

$$
\begin{aligned}
f(x) =& f(0) + f^{'}(0)x + \frac{f^{''}(0)x^2}{2!} + \ldots + \frac{f^k(0)x^k}{k!} + \ldots\\
=& x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\\
=& \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}.
\end{aligned}
$$

Example: Find the Manclaurin series expansion of $f(x) = cos(x)$?

We have

$$
\begin{aligned}
f(x) = cos(x) &\Longrightarrow f(0) = cos(0) = 1,\\
f'(x) = -sin(x) &\Longrightarrow f^{'}(0) = -sin(0) = 0,\\
f''(x) = -cos(x) &\Longrightarrow f^{''}(0) = -cos(0) = -1,\\
f'''(x) = sin(x) &\Longrightarrow f^{'''}(0) = sin(0) = 0,\\
f^4(x) = cos(x) &\Longrightarrow f^4(0) = cos(0) = 1,
\end{aligned}
$$

and so on. Thus, the Manclaurin series is

$$
\begin{aligned}
f(x) =& f(0) + f^{'}(0)x + \frac{f^{''}(0)x^2}{2!} + \ldots + \frac{f^k(0)x^k}{k!} + \ldots\\
=& 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\\
=& \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}.
\end{aligned}
$$

Note that if $f(x) = sin(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$, then

$$
\begin{aligned}
f^{'}(x) =& \sum_{n=0}^{\infty}(-1)^n \frac{(2n+1)x^{2n+1-1}}{(2n+1)!}\\
=& \sum_{n=0}^{\infty}(-1)^n \frac{(2n+1)x^{2n}}{(2n+1)(2n)!}\\
=& \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}\\
=& cos(x).
\end{aligned}
$$

Find the Manclaurin series expansion of $f(x) = e^x$?

Exercise

Example: Express $\int sin(x^2)dx$ as a series?

Note that

$$\begin{aligned} sin(x) =& x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \\ sin(x^2) =& x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \dots \\ =& x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots \end{aligned}$$

So, we have

$$\begin{aligned} \int sin(x^2)dx =& \int \Big[x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots\Big]dx \\ =& \frac{x^3}{3} - \frac{x^7}{7 \times 3!} + \frac{x^{11}}{11 \times 5!} - \frac{x^{15}}{15 \times 7!} + \dots \end{aligned}$$

Example: show that $e^{i\theta} = cos(\theta) + isin(\theta)$, where $i = \sqrt{-1}$ is complex number by using power series?

Note that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$

Also, we have $i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i, i^6 = -1$, and so on.

$$\begin{aligned} e^{i\theta} =& 1 + \frac{(i\theta)}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \dots \\ =& 1 + \frac{i\theta}{1!} - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} + \dots \\ =& \Big(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots\Big) + i\Big(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots\Big) \\ =& \sum_{n=0}^{\infty}(-1)^n \frac{\theta^{2n}}{(2n)!} + i\sum_{n=0}^{\infty}(-1)^n \frac{\theta^{2n+1}}{(2n+1)!} \\ =& cos(\theta) + isin(\theta). \end{aligned}$$

Finite series

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n .$$

$$\boldsymbol{Ex1}: \sum_{k=1}^{8} k = 1 + 2 + 3 + \cdots + 7 + 8$$

$$\boldsymbol{Ex2}: \sum_{k=1}^{25} k^2 = 1^2 + 2^2 + 3^2 + \cdots + 24^2 + 25^2$$

Properties

1. *Sum Rule:* $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$
2. *Difference Rule:* $\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$
3. *Constant Multiple Rule:* $\sum_{k=1}^{n} ca_k = c \cdot \sum_{k=1}^{n} a_k$ (Any number c)
4. *Constant Value Rule:* $\sum_{k=1}^{n} c = n \cdot c$ (c is any constant value.)

Formulas for the sums :

1) $\sum_{k=1}^{n} c = n.c$

2) $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

3) $\sum_{k=1}^{n} k^2 = \frac{n(1+n)(2n+1)}{6}$

4) $\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$

To prove (1) H.W

To prove (2), the formula in general, we write out the terms in the sum twice, once forward and once backward.

$$\sum_{k=1}^{n} k = 1+2+3+\cdots+n-2+n-1+n$$

$$\sum_{k=1}^{n} k = n+n-1+n-2+\cdots+3+2+1$$

If we add the two terms we get

$$2\sum_{k=1}^{n} k = \underbrace{(1+n)+(1+n)+(1+n)+\cdots+(1+n)}_{n-times}$$

$$2\sum_{k=1}^{n} k = n\,(1+n) \Longrightarrow \sum_{k=1}^{n} k = \frac{n(1+n)}{2}$$

Now, we want to prove (3) we will use the fact

$$(k+1)^3 - k^3 = 3k^2 + 3k + 1 \cdots (*)$$

Note that

$$\sum_{k=1}^{n}[(k+1)^3 - k^3] =$$

$$[2^3-1^3]+[3^3-2^3]+[4^3-3^3]+\cdots+[n^3-(n-1)^3]+\ [(n+1)^3-n^3]$$

$$[\cancel{2^3}-1^3]+[\cancel{3^3}-\cancel{2^3}]+[\cancel{4^3}-\cancel{3^3}]+\cdots+[\cancel{n^3}-(n-\cancel{1})^3]+\ [(n+1)^3-\cancel{n^3}]$$

$$\sum_{k=1}^{n}[(k+1)^3 - k^3] = (n+1)^3 - 1$$

Now, taking the sum for both sides of $(*)$

$$\sum_{k=1}^{n}[(k+1)^3-k^3]=\sum_{k=1}^{n}[3k^2+3k+1]$$

$$\sum_{k=1}^{n}[(k+1)^3-k^3]=3\sum_{k=1}^{n}k^2+3\sum_{k=1}^{n}k+\sum_{k=1}^{n}1$$

$$(n+1)^3-1=3\sum_{k=1}^{n}k^2+\frac{3n(1+n)}{2}+n$$

$$3\sum_{k=1}^{n}k^2=(n+1)^3-1-\frac{3n(1+n)}{2}-n$$

$$3\sum_{k=1}^{n}k^2=(n+1)^3-(1+n)-\frac{3n(1+n)}{2}$$

$$3\sum_{k=1}^{n}k^2=(1+n)[(n+1)^2-1-\frac{3n}{2}]$$

$$3\sum_{k=1}^{n}k^2=(1+n)[n^2+2n+1-1-\frac{3n}{2}]$$

$$3\sum_{k=1}^{n}k^2=(1+n)[n^2+\frac{n}{2}]$$

$$3\sum_{k=1}^{n}k^2=(1+n)(\frac{2n^2+n}{2})$$

$$\sum_{k=1}^{n}k^2=\frac{n(1+n)(2n+1)}{6}$$

To prove (4) H.W